

Mould expansions for the saddle-node and resurgence monomials

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Abstract. This article is an introduction to some aspects of Écalle's mould calculus, a powerful combinatorial tool which yields surprisingly explicit formulas for the normalising series attached to an analytic germ of singular vector field or of map. This is illustrated on the case of the saddle-node, a two-dimensional vector field which is formally conjugate to Euler's vector field $x^2 \frac{\partial}{\partial x} + (x+y) \frac{\partial}{\partial y}$, and for which the formal normalisation is shown to be resurgent in $1/x$. Resurgence monomials adapted to alien calculus are also described as another application of mould calculus.

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1 Introduction

Mould calculus was developed by J. Écalle in relation with his Resurgence theory almost thirty years ago ([3], [6], [7]). The primary goal of this text is to give an introduction to mould calculus, together with an exposition of the way it can be applied to a specific geometric problem pertaining to the theory of dynamical systems: the analytic classification of saddle-node singularities.

The treatment of this example was indicated in [4] in concise manner (see also [2]), but I found it useful to provide a self-contained presentation of mould calculus and detailed explanations for the saddle-node problem, in the same spirit as Resurgence theory and alien calculus were presented in [14] together with the example of the analytic classification of tangent-to-identity transformations in complex dimension 1.

Basic facts from Resurgence theory are also recalled in the course of the exposition, with the hope that this text will serve to a broad readership. I also included a section on the relation between the resurgent approach to the saddle-node problem and Martinet-Ramis's work [12].

The text consists of three parts.

- A. Section 2 describes the problem of the normalisation of the saddle-node and Section 3 outlines its treatment by the method of mould-comould expansions.
- B. The second part has an “algebraic” flavour: it is devoted to a systematic exposition of some features of mould algebras (Sections 4 and 5) and mould-comould expansions (Sections 6 and 7).
- C. The third part is mainly concerned by the applications to Resurgence theory of the previous results (Sections 8–11 show the consequences for the problem of the saddle-node and have an “analytic” flavour, Section 12 describes the construction of resurgence monomials which allow one to check the freeness of the algebra of alien derivations); other applications are also briefly alluded to in Section 13 (with a few words about arborification and multizetas).

All the ideas come from J. Écalle's articles and lectures. An effort has been made to provide full details, which occasionally may have resulted in original definitions, but they must be considered as auxiliary with respect to the overall theory. The details of the resurgence proofs which are given in Sections 8 and 10 are original, at least I did not see them in the literature previously.

Part A: The saddle-node problem

2 The saddle-node and its formal normalisation

2.1 Let us consider a germ of complex analytic 2-dimensional vector field

$$X = x^2 \frac{\partial}{\partial x} + A(x, y) \frac{\partial}{\partial y}, \quad A \in \mathbb{C}\{x, y\}, \quad (2.1)$$

for which we assume

$$A(0, y) = y, \quad \frac{\partial^2 A}{\partial x \partial y}(0, 0) = 0. \quad (2.2)$$

Assumption (2.2) ensures that X is formally conjugate to the normal form

$$X_0 = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \quad (2.3)$$

We shall be interested in the formal transformations which conjugate X and X_0 .

2.2 This is the simplest case from the point of view of *formal* classification of saddle-node singularities of analytic differential equations.¹ Indeed, when a differential equation $B(x, y) dy - A(x, y) dx = 0$ is singular at the origin ($A(0, 0) = B(0, 0) = 0$) and its 1-jet has eigenvalues 0 and 1, it is always formally conjugate to one of the normal forms $x^{p+1} dy - (1 + \lambda x)y dx = 0$ ($p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$) or $y dx = 0$. What we call saddle-node singularity corresponds to the first case, and the normal form X_0 corresponds to $p = 1$ and $\lambda = 0$.

Moreover, a saddle-node singularity can always be analytically reduced to the form $x^{p+1} dy - A(x, y) dx = 0$ with $A(0, y) = y$ (this result goes back to Dulac—see [12], [13]), it is thus legitimate to consider vector fields of the form (2.1), which generate the same foliations (we restrict ourselves to $(p, \lambda) = (1, 0)$ for the sake of simplicity).

The problem of the *analytic* classification of saddle-node singularities was solved in [12]. The resurgent approach to this problem is indicated in [4] and [3, Vol. 3] (see also [2]). The resurgent approach consists in analysing the divergence of the normalising transformation through alien calculus.

2.3 Normalising transformation means a formal diffeomorphism θ solution of the conjugacy equation

$$X = \theta^* X_0. \quad (2.4)$$

Due to the shape of X , one can find a unique formal solution of the form

$$\theta(x, y) = (x, \varphi(x, y)), \quad \varphi(x, y) = y + \sum_{n \geq 0} \varphi_n(x) y^n, \quad \varphi_n(x) \in x\mathbb{C}[[x]]. \quad (2.5)$$

¹A singular differential equation is essentially the same thing as a differential 1-form which vanish at the origin. It defines a singular foliation, the leaves of which can also be obtained by integrating a singular vector field, but classifying singular foliations (or singular differential equations) is equivalent to classifying singular vector fields *up to time-change*. See e.g. [13].

The first step in the resurgent approach consists in proving that the formal series φ_n are resurgent with respect to the variable $z = -1/x$. We shall prove this fact by using Écalle's mould calculus (see Theorem 2 in Section 8 below).

The Euler series $\varphi_0(x) = -\sum_{n \geq 1} (n-1)! x^n$ appears in the case $A(x, y) = x + y$, for which the solution of the conjugacy equation is simply $\theta(x, y) = (x, y + \varphi_0(x))$.

2.4 Observe that $\theta(x, y) = \left(x, y + \sum_{n \geq 0} \varphi_n(x) y^n \right)$ is solution of the conjugacy equation if and only if

$$\tilde{Y}(z, u) = u e^z + \sum_{n \geq 0} u^n e^{nz} \tilde{\varphi}_n(z), \quad \tilde{\varphi}_n(z) = \varphi_n(-1/z) \in z^{-1} \mathbb{C}[[z^{-1}]], \quad (2.6)$$

is solution of the differential equation

$$\partial_z \tilde{Y} = A(-1/z, \tilde{Y}) \quad (2.7)$$

associated with the vector field X . (Indeed, the first component of the flow of X is trivial and the second component is determined by solving (2.7); on the other hand, the flow of X_0 is trivial and, by plugging it into θ , one obtains the flow of X .)

The formal expansion $\tilde{Y}(z, u)$ is called *formal integral* of the differential equation (2.7). One can obtain its components $\tilde{\varphi}_n(z)$ (and, consequently, the formal series $\varphi_n(x)$ themselves) as solutions of ordinary differential equations, by expanding (2.7) in powers of u :

$$\frac{d\tilde{\varphi}_0}{dz} = A(-1/z, \tilde{\varphi}_0(z)), \quad (2.8)$$

$$\frac{d\tilde{\varphi}_n}{dz} + n\tilde{\varphi}_n(z) = \partial_y A(-1/z, \tilde{\varphi}_0(z))\tilde{\varphi}_n(z) + \tilde{\chi}_n(z), \quad (2.9)$$

with $\tilde{\chi}_n$ inductively determined by $\tilde{\varphi}_0, \dots, \tilde{\varphi}_{n-1}$. Only the first equation is non-linear. One can prove the resurgence of the $\tilde{\varphi}_n$'s by exploiting their property of being the unique solutions in $z^{-1} \mathbb{C}[[z^{-1}]]$ of these equations and devising a perturbative scheme to solve the first one,² but mould calculus is quite a different approach.

3 Mould-comould expansions for the saddle-node

3.1 The analytic vector fields X and X_0 can be viewed as derivations of the algebra $\mathbb{C}\{x, y\}$, but since we are interested in formal conjugacy, we now consider them as derivations of $\mathbb{C}[[x, y]]$. We shall first rephrase our problem as a problem about operators of this algebra.³

²See Section 2.1 of [14] for an illustration of this method on a non-linear difference equation.

³Our algebras will be, unless otherwise specified, associative unital algebras over \mathbb{C} (possibly non-commutative). In this article, operator means endomorphism of the underlying vector space; thus an operator of $\mathcal{A} = \mathbb{C}[[x, y]]$ is an element of $\text{End}_{\mathbb{C}}(\mathcal{A})$. The space $\text{End}_{\mathbb{C}}(\mathcal{A})$ has natural

The commutative algebra $\mathcal{A} = \mathbb{C}[[x, y]]$ is also a local ring; as such, it is endowed with a metrizable topology, in which the powers of the maximal ideal $\mathfrak{M} = \{ f \in \mathbb{C}[[x, y]] \mid f(0, 0) = 0 \}$ form a system of neighbourhoods of 0, which we call Krull topology or topology of the formal convergence and which is complete (as a uniform structure).⁴

Lemma 3.1. *The set of all continuous algebra homomorphisms of $\mathbb{C}[[x, y]]$ coincides with the set of all substitution operators, i.e. operators of the form $f \mapsto f \circ \theta$ with $\theta \in \mathfrak{M} \times \mathfrak{M}$.*

Proof. Any substitution operator is clearly a continuous algebra homomorphism of $\mathbb{C}[[x, y]]$. Conversely, let Θ be a continuous algebra homomorphism. The idea is that Θ will be determined by its action on the two generators of the maximal ideal, and setting $\theta = (\Theta x, \Theta y)$ we can identify Θ with the substitution operator $f \mapsto f \circ \theta$. We just need to check that Θx and Θy both belong to the maximal ideal, which is the case because, by continuity, $(\Theta x)^n = \Theta(x^n)$ and $(\Theta y)^n = \Theta(y^n)$ must tend to 0 as $n \rightarrow \infty$; one can then write any f as a convergent—for the Krull topology—series of monomials $\sum f_{m,n} x^m y^n$ and its image as the formally convergent series $\Theta f = \sum \Theta(f_{m,n} x^m y^n) = \sum f_{m,n} (\Theta x)^m (\Theta y)^n$. \square

A formal invertible transformation thus amounts to a continuous automorphism of $\mathbb{C}[[x, y]]$. Since the conjugacy equation (2.4) can be written

$$Xf = [X_0(f \circ \theta)] \circ \theta^{-1}, \quad f \in \mathbb{C}[[x, y]],$$

if we work at the level of the substitution operator, we are left with the problem of finding a continuous automorphism Θ of $\mathbb{C}[[x, y]]$ such that $\Theta(Xf) = X_0(\Theta f)$ for all f , i.e.

$$\Theta X = X_0 \Theta. \tag{3.1}$$

3.2 The idea is to construct a solution to (3.1) from the “building blocks” of X . Let us use the Taylor expansion

$$A(x, y) = y + \sum_{n \in \mathcal{N}} a_n(x) y^{n+1}, \quad \mathcal{N} = \{ n \in \mathbb{Z} \mid n \geq -1 \} \tag{3.2}$$

to write

$$X = X_0 + \sum_{n \in \mathcal{N}} a_n(x) B_n, \quad B_n = y^{n+1} \frac{\partial}{\partial y}, \tag{3.3}$$

$$a_n(x) \in x\mathbb{C}\{x\}, \quad a_0(x) \in x^2\mathbb{C}\{x\} \tag{3.4}$$

(thus incorporating the information from (2.2)). The series in (3.3) must be interpreted as a simply convergent series of operators of $\mathbb{C}[[x, y]]$ (the series $\sum a_n B_n f$ is formally convergent for any $f \in \mathbb{C}[[x, y]]$).

structures of ring and of \mathcal{A} -module, which are compatible in the sense that $f \in \mathcal{A} \mapsto f \text{Id} \in \text{End}_{\mathbb{C}}(\mathcal{A})$ is a ring homomorphism.

⁴This is also called the \mathfrak{M} -adic topology, or the (x, y) -adic topology. Beware that $\mathbb{C}[[x, y]]$ is a topological algebra only if we put the discrete topology on \mathbb{C} .

Let us introduce the differential operators

$$\mathbf{B}_\emptyset = \text{Id}, \quad \mathbf{B}_{\omega_1, \dots, \omega_r} = B_{\omega_r} \cdots B_{\omega_1} \quad (3.5)$$

for $\omega_1, \dots, \omega_r \in \mathcal{N}$. We shall look for an automorphism Θ solution of (3.1) in the form

$$\Theta = \sum_{r \geq 0} \sum_{\omega_1, \dots, \omega_r \in \mathcal{N}} \mathcal{V}^{\omega_1, \dots, \omega_r}(x) \mathbf{B}_{\omega_1, \dots, \omega_r}, \quad (3.6)$$

with the convention that the only term with $r = 0$ is $\mathcal{V}^\emptyset \mathbf{B}_\emptyset$, with $\mathcal{V}^\emptyset = 1$, and with coefficients $\mathcal{V}^{\omega_1, \dots, \omega_r}(x) \in x\mathbb{C}[[x]]$ to be determined from the data $\{a_n, n \in \mathcal{N}\}$ in such a way that

- (i) the expression (3.6) is a formally convergent series of operators of $\mathbb{C}[[x, y]]$ and defines an operator Θ which is continuous for the Krull topology,
- (ii) the operator Θ is an algebra automorphism,
- (iii) the operator Θ satisfies the conjugacy equation (3.1).

In Écalle's terminology, the collection of operators $\{\mathbf{B}_{\omega_1, \dots, \omega_r}\}$ is a typical example of *comould*; any collection of coefficients $\{\mathcal{V}^{\omega_1, \dots, \omega_r}\}$ is a *mould* (here with values in $\mathbb{C}[[x]]$, but other algebras may be used); a formally convergent series of the form (3.6) is a *mould-comould expansion*, often abbreviated as

$$\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$$

(we shall clarify later what "formally convergent" means for such multiply-indexed series of operators).

3.3 Let us indicate right now the formulas for the problem of the saddle-node (2.1):

Lemma 3.2. *The equations*

$$(x^2 \frac{d}{dx} + \omega_1 + \cdots + \omega_r) \mathcal{V}^{\omega_1, \dots, \omega_r} = a_{\omega_1} \mathcal{V}^{\omega_2, \dots, \omega_r}, \quad \omega_1, \dots, \omega_r \in \mathcal{N} \quad (3.7)$$

inductively determine a unique collection of formal series $\mathcal{V}^{\omega_1, \dots, \omega_r} \in x\mathbb{C}[[x]]$ for $r \geq 1$. Moreover,

$$\mathcal{V}^{\omega_1, \dots, \omega_r} \in x^{\lceil r/2 \rceil} \mathbb{C}[[x]], \quad (3.8)$$

where $\lceil s \rceil$ denotes, for any $s \in \mathbb{R}$, the least integer not smaller than s .

Proof. Let ν denote the valuation in $\mathbb{C}[[x]]$: $\nu(\sum c_m x^m) = \min\{m \mid c_m \neq 0\} \in \mathbb{N}$ for a non-zero formal series and $\nu(0) = \infty$.

Since $\partial = x^2 \frac{d}{dx}$ increases valuation by at least one unit, $\partial + \mu$ is invertible for any $\mu \in \mathbb{C}^*$ and the inverse operator

$$(\partial + \mu)^{-1} = \sum_{r \geq 0} \mu^{-r-1} (-\partial)^r \quad (3.9)$$

(formally convergent series of operators) leaves $x\mathbb{C}[[x]]$ invariant. On the other hand, we define $\partial^{-1}: x^2 \mathbb{C}[[x]] \rightarrow x\mathbb{C}[[x]]$ by the formula $\partial^{-1}\varphi(x) = \int_0^x (t^{-2}\varphi(t)) dt$,

so that $\psi = \partial^{-1}\varphi$ is the unique solution in $x\mathbb{C}[[x]]$ of the equation $\partial\psi = \varphi$ whenever $\varphi \in x^2\mathbb{C}[[x]]$.

For $r = 1$, equation (3.7) has a unique solution \mathcal{V}^{ω_1} in $x\mathbb{C}[[x]]$, because the right-hand side is a_{ω_1} , element of $x\mathbb{C}[[x]]$, and even of $x^2\mathbb{C}[[x]]$ when $\omega_1 = 0$. By induction, for $r \geq 2$, we get a right-hand side in $x^2\mathbb{C}[[x]]$ and a unique solution \mathcal{V}^ω in $x\mathbb{C}[[x]]$ for $\omega = (\omega_1, \dots, \omega_r) \in \mathcal{N}^r$. Moreover, with the notation ' $\omega = (\omega_2, \dots, \omega_r)$ ', we have

$$\nu(\mathcal{V}^\omega) \geq \alpha^\omega + \nu(\mathcal{V}^\omega), \quad \text{with } \alpha^\omega = \begin{cases} 0 & \text{if } \omega_1 + \dots + \omega_r = 0 \text{ and } \omega_1 \neq 0, \\ 1 & \text{if } \omega_1 + \dots + \omega_r \neq 0 \text{ or } \omega_1 = 0. \end{cases}$$

Thus $\nu(\mathcal{V}^\omega) \geq \text{card } \mathcal{R}^\omega$, with $\mathcal{R}^\omega = \{ i \in [1, r] \mid \omega_i + \dots + \omega_r \neq 0 \text{ or } \omega_i = 0 \}$ for $r \geq 1$.

Let us check that $\text{card } \mathcal{R}^\omega \geq \lceil r/2 \rceil$. This stems from the fact that if $i \notin \mathcal{R}^\omega$, $i \geq 2$, then $i - 1 \in \mathcal{R}^\omega$ (indeed, in that case $\omega_{i-1} + \dots + \omega_r = \omega_{i-1}$), and that \mathcal{R}^ω has at least one element, namely r . The inequality is thus true for $r = 1$ or 2 ; by induction, if $r \geq 3$, then $\mathcal{R}^\omega \cap [3, r] = \mathcal{R}^{\omega'} \cup \{r\}$ with ' $\omega' = (\omega_3, \dots, \omega_r)$ ' and either $2 \in \mathcal{R}^{\omega'}$, or $2 \notin \mathcal{R}^{\omega'}$ and $1 \in \mathcal{R}^{\omega'}$, thus $\text{card } \mathcal{R}^\omega \geq 1 + \text{card } \mathcal{R}^{\omega'}$. \square

3.4 To give a definition of formally summable families of operators adapted to our needs, we shall consider our operators as elements of a topological ring of a certain kind and make use of the Cauchy criterium for summable families.

Definition 3.1. Given a ring \mathcal{E} (possibly non-commutative), we call pseudovaluation any map $\text{val}: \mathcal{E} \rightarrow \mathbb{Z} \cup \{\infty\}$ satisfying, for any $\Theta, \Theta_1, \Theta_2 \in \mathcal{E}$,

- $\text{val}(\Theta) = \infty$ iff $\Theta = 0$,
- $\text{val}(\Theta_1 - \Theta_2) \geq \min \{ \text{val}(\Theta_1), \text{val}(\Theta_2) \}$,
- $\text{val}(\Theta_1 \Theta_2) \geq \text{val}(\Theta_1) + \text{val}(\Theta_2)$,

The formula $d_{\text{val}}(\Theta_1, \Theta_2) = 2^{-\text{val}(\Theta_2 - \Theta_1)}$ then defines a distance, for which \mathcal{E} is a topological ring. We call $(\mathcal{E}, \text{val})$ a complete pseudovaluation ring if the distance d_{val} is complete.

We use the word pseudovaluation rather than valuation because \mathcal{E} is not assumed to be an integral domain, and we do not impose equality in the third property. The distance d_{val} is ultrametric, translation-invariant, and it satisfies $d_{\text{val}}(0, \Theta_1 \Theta_2) \leq d_{\text{val}}(0, \Theta_1) d_{\text{val}}(0, \Theta_2)$.

Let us denote by 1 the unit of \mathcal{E} . Giving a pseudovaluation on \mathcal{E} such that $\text{val}(1) = 0$ is equivalent to giving a filtration $(\mathcal{E}_\delta)_{\delta \in \mathbb{Z}}$ that is compatible with its ring structure (i.e. a sequence of additive subgroups such that $1 \in \mathcal{E}_0$, $\mathcal{E}_{\delta+1} \subset \mathcal{E}_\delta$ and $\mathcal{E}_\delta \mathcal{E}_{\delta'} \subset \mathcal{E}_{\delta+\delta'}$ for all $\delta, \delta' \in \mathbb{Z}$), exhaustive ($\bigcup \mathcal{E}_\delta = \mathcal{E}$) and separated ($\bigcap \mathcal{E}_\delta = \{0\}$). Indeed, the order function val associated with the filtration, defined by $\text{val}(\Theta) = \sup \{ \delta \in \mathbb{Z} \mid \Theta \in \mathcal{E}_\delta \}$, is then a pseudovaluation; conversely, one can set $\mathcal{E}_\delta = \{ \Theta \in \mathcal{E} \mid \text{val}(\Theta) \geq \delta \}$.

Definition 3.2. Let $(\mathcal{E}, \text{val})$ be a complete pseudovaluation ring. Given a set I , a family $(\Theta_i)_{i \in I}$ in \mathcal{E} is said to be formally summable if, for any $\delta \in \mathbb{Z}$, the set

$\{ i \in I \mid \text{val}(\Theta_i) \leq \delta \}$ is finite (the support of the family is thus countable, if not I itself).

One can then check that, for any exhaustion $(I_k)_{k \in \mathbb{N}}$ by finite sets of the support of the family, the sequence $\sum_{i \in I_k} \Theta_i$ is a Cauchy sequence for d_{val} , and that the limit does not depend on the chosen exhaustion; the common limit is then denoted $\sum_{i \in I} \Theta_i$. Observe that there must exist $\delta_* \in \mathbb{Z}$ such that $\text{val}(\Theta_i) \geq \delta_*$ for all $i \in I$.

3.5 We apply this to operators of $\mathcal{A} = \mathbb{C}[[x, y]]$ as follows. The Krull topology of \mathcal{A} can be defined with the help of the monomial valuation

$$\nu_4(f) = \min\{4m + n \mid f_{m,n} \neq 0\} \quad \text{for } f = \sum f_{m,n} x^m y^n \neq 0, \quad \nu_4(0) = \infty.$$

Indeed, for any sequence $(f_k)_{k \in \mathbb{N}}$ of \mathcal{A} ,

$$f_k \xrightarrow[k \rightarrow \infty]{} 0 \iff \sum_{k \in \mathbb{N}} f_k \text{ formally convergent} \iff \nu_4(f_k) \xrightarrow[k \rightarrow \infty]{} \infty.$$

In particular, $(\mathbb{C}[[x, y]], \nu_4)$ is a complete pseudovaluation ring.

Suppose more generally that (\mathcal{A}, ν) is any complete pseudovaluation ring such that \mathcal{A} is also an algebra. Corresponding to the filtration $\mathcal{A}_p = \{f \in \mathcal{A} \mid \nu(f) \geq p\}$, $p \in \mathbb{Z}$, there is a filtration of $\text{End}_{\mathbb{C}}(\mathcal{A})$:

$$\mathcal{E}_\delta = \{\Theta \in \text{End}_{\mathbb{C}}(\mathcal{A}) \mid \Theta(\mathcal{A}_p) \subset \mathcal{A}_{p+\delta} \text{ for each } p\}, \quad \delta \in \mathbb{Z}.$$

Definition 3.3. Let $\delta \in \mathbb{Z}$. An element Θ of \mathcal{E}_δ is said to be an “operator of valuation $\geq \delta$ ”. We then define $\text{val}_\nu(\Theta) \in \mathbb{Z} \cup \{\infty\}$, the “valuation of Θ ”, as the largest δ_0 such that Θ has valuation $\geq \delta_0$; this number is infinite only for $\Theta = 0$.

Denote by \mathcal{E} the union $\bigcup \mathcal{E}_\delta$ over all $\delta \in \mathbb{Z}$: these are the operators of \mathcal{A} “having a valuation” (with respect to ν), i.e.

$$\mathcal{E} = \{\Theta \in \text{End}_{\mathbb{C}}(\mathcal{A}) \mid \text{val}_\nu(\Theta) = \inf_{f \in \mathcal{A}} \{\nu(\Theta f) - \nu(f)\} > -\infty\}.$$

They clearly are continuous for the topology induced by ν on \mathcal{A} ; they form a subalgebra of the algebra of all continuous operators⁵ and $(\mathcal{E}, \text{val}_\nu)$ is a complete pseudovaluation ring. For any formally summable family $(\Theta_i)_{i \in I}$ of sum Θ in \mathcal{E} and $f \in \mathcal{A}$, the family $(\Theta_i f)_{i \in I}$ is summable in the topological ring \mathcal{A} , with sum Θf .

Lemma 3.3. *With the notation of formula (3.5) and Lemma 3.2, the family $(\mathcal{V}^{\omega_1, \dots, \omega_r} \mathbf{B}_{\omega_1, \dots, \omega_r})_{r \geq 1, \omega_1, \dots, \omega_r \in \mathbb{N}}$ is formally summable in the algebra of operators of $\mathbb{C}[[x, y]]$ having a valuation with respect to ν_4 . In particular the resulting operator Θ is continuous for the Krull topology. Similarly, the formula*

$$\mathcal{V}^{\omega_1, \dots, \omega_r} = (-1)^r \mathcal{V}^{\omega_r, \dots, \omega_1} \tag{3.10}$$

gives rise to a formally summable family $(\mathcal{V}^{\omega_1, \dots, \omega_r} \mathbf{B}_{\omega_1, \dots, \omega_r})_{r \geq 1, \omega_1, \dots, \omega_r \in \mathbb{N}}$.

⁵Not all continuous operators of \mathcal{A} belong to \mathcal{E} : think of the operator of $\mathbb{C}[[y]]$ which maps y^m to $y^{m/2}$ if m is even and to 0 if m is odd.

Proof. Clearly $\nu_4(B_n f) \geq \nu_4(f) + n$ and, by induction,

$$\nu_4(\mathbf{B}_{\omega_1, \dots, \omega_r} f) \geq \nu_4(f) + \omega_1 + \dots + \omega_r.$$

As a consequence of (3.8),

$$\nu_4(\mathcal{V}^{\omega_1, \dots, \omega_r} \mathbf{B}_{\omega_1, \dots, \omega_r} f) \geq \nu_4(f) + \omega_1 + \dots + \omega_r + 2r, \quad \omega_1, \dots, \omega_r \in \mathbb{N}.$$

Hence, with the above notations, each $\mathcal{V}^{\omega_1, \dots, \omega_r} \mathbf{B}_{\omega_1, \dots, \omega_r}$ is an element \mathcal{E} with valuation $\geq \omega_1 + \dots + \omega_r + 2r$, and the same thing holds for each $\mathcal{V}^{\omega_1, \dots, \omega_r} \mathbf{B}_{\omega_1, \dots, \omega_r}$.

The ω_i 's may be negative but they are always ≥ -1 , thus $\omega_1 + \dots + \omega_r + r \geq 0$. Therefore, for any $\delta > 0$, the condition $\omega_1 + \dots + \omega_r + 2r \leq \delta$ implies $r \leq \delta$ and $\sum(\omega_i + 1) = \omega_1 + \dots + \omega_r + r \leq \delta$. Since this condition is fulfilled only a finite number of times, the conclusion follows. \square

3.6 Here is the key statement, the proof of which will be spread over Sections 4–7:

Theorem 1. *The continuous operator $\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$ defined by Lemmas 3.2 and 3.3 is an algebra automorphism of $\mathbb{C}[[x, y]]$ which satisfies the conjugacy equation (3.1). The inverse operator is $\sum \mathcal{V}^\bullet \mathbf{B}_\bullet$.*

Observe that $\Theta x = x$, thus Θ is must be the substitution operator for a formal transformation of the form $\theta(x, y) = (x, \varphi(x, y))$, with $\varphi = \Theta y$, in accordance with (2.5). An easy induction yields

$$\mathbf{B}_\omega y = \beta_\omega y^{\omega_1 + \dots + \omega_r + 1}, \quad \omega \in \mathbb{N}^r, r \geq 1, \quad (3.11)$$

with $\beta_\omega = 1$ if $r = 1$, $\beta_\omega = (\omega_1 + 1)(\omega_1 + \omega_2 + 1) \dots (\omega_1 + \dots + \omega_{r-1} + 1)$ if $r \geq 2$. We have $\beta_\omega = 0$ whenever $\omega_1 + \dots + \omega_r \leq -2$ (since (3.11) holds a priori in the fraction field $\mathbb{C}((y))$ but $\mathbf{B}_\omega y$ belongs to $\mathbb{C}[[y]]$), hence

$$\theta(x, y) = (x, \varphi(x, y)),$$

$$\varphi(x, y) = y + \sum_{n \geq 0} \varphi_n(x) y^n, \quad \varphi_n = \sum_{\substack{r \geq 1, \omega \in \mathbb{N}^r \\ \omega_1 + \dots + \omega_r + 1 = n}} \beta_\omega \mathcal{V}^\omega \quad (3.12)$$

(in the series giving φ_n , there are only finitely many terms for each r , (3.8) thus yields its formal convergence in $x\mathbb{C}[[x]]$).

Similarly, $\Theta^{-1} = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$ is the substitution operator of a formal transformation $(x, y) \mapsto (x, \psi(x, y))$, which is nothing but θ^{-1} , and

$$\psi(x, y) = \Theta^{-1} y = y + \sum_{n \geq 0} \psi_n(x) y^n, \quad (3.13)$$

where each coefficient can be represented as a formally convergent series $\psi_n = \sum_{\omega_1 + \dots + \omega_r + 1 = n} \beta_\omega \mathcal{V}^\omega$.

See Lemma 8.6 on p. 45 for formulas relating directly the φ_n 's and the ψ_n 's.

Remark 3.1. The \mathcal{V}^ω 's are generically divergent with at most Gevrey-1 growth of the coefficients, as can be expected from formula (3.9); for instance, for $\omega_1 \neq 0$,

we get $\mathcal{V}^{\omega_1}(x) = \sum \omega_1^{-r-1} (-x^2 \frac{d}{dx})^r a_{\omega_1}$ which is generically divergent because the repeated differentiations are not compensated by the division by any factorial-like expression. This divergence is easily studied through formal Borel transform with respect to $z = -1/x$, which is the starting point of the resurgent analysis of the saddle-node—see Section 8. We shall see in Section 9 why the \mathcal{V}^ω 's can be called “resurgence monomials”.

The proof of Theorem 1 will follow easily from the general notions introduced in the next sections.

Part B: The formalism of moulds

4 The algebra of moulds

4.1 In this section and the next three ones, we assume that we are given a non-empty set Ω and a commutative \mathbb{C} -algebra \mathbf{A} , the unit of which is denoted 1. In the previous section, the roles of Ω and \mathbf{A} were played by \mathcal{N} and $\mathbb{C}[[x]]$.

It is sometimes convenient to have a commutative semigroup structure on Ω ; then we would rather take $\Omega = \mathbb{Z}$ in the previous section and consider that the mould $\{\mathcal{V}^{\omega_1, \dots, \omega_r}\}$ was defined on \mathbb{Z} but supported on \mathcal{N} (i.e. we extend it by 0 whenever one of the ω_i 's is ≤ -2).

We consider Ω as an alphabet and denote by Ω^\bullet the free monoid of *words*: a word is any finite sequence of letters, $\omega = (\omega_1, \dots, \omega_r)$ with $\omega_1, \dots, \omega_r \in \Omega$; its *length* $r = r(\omega)$ can be any non-negative integer. The only word of zero length is the empty word, denoted \emptyset , which is the unit of *concatenation*, the monoid law $(\omega, \eta) \mapsto \omega \cdot \eta$ defined by

$$(\omega_1, \dots, \omega_r) \cdot (\eta_1, \dots, \eta_s) = (\omega_1, \dots, \omega_r, \eta_1, \dots, \eta_s)$$

for non-empty words.

As previously alluded to, a *mould on Ω with values in \mathbf{A}* is nothing but a map $\Omega^\bullet \rightarrow \mathbf{A}$. It is customary to denote the value of the mould on a word ω by affixing ω as an upper index to the symbol representing the mould, and to refer to the mould itself by using \bullet as upper index. Hence \mathcal{V}^\bullet is the mould, the value of which at ω is denoted \mathcal{V}^ω .

A mould with values in \mathbb{C} is called a *scalar mould*.

4.2 Being the set of all maps from a set to the ring \mathbf{A} , the set of moulds $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ has a natural structure of \mathbf{A} -module: addition and ring multiplication are defined component-wise (for instance, if $\mu \in \mathbf{A}$ and $M^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$, the mould $N^\bullet = \mu M^\bullet$ is defined by $N^\omega = \mu M^\omega$ for all $\omega \in \Omega^\bullet$).

The ring structure of \mathbf{A} together with the monoid structure of Ω^\bullet also give rise to a *multiplication of moulds*, thus defined:

$$P^\bullet = M^\bullet \times N^\bullet: \omega \mapsto P^\omega = \sum_{\omega=\omega^1+\omega^2} M^{\omega^1} N^{\omega^2}, \quad (4.1)$$

with summation over the $r(\omega) + 1$ decompositions of ω into two words (including ω^1 or $\omega^2 = \emptyset$). Mould multiplication is associative but not commutative (except if Ω has only one element). We get a ring structure on $\mathcal{M}^\bullet(\Omega, \mathbf{A})$, with unit

$$1^\bullet: \omega \mapsto 1^\omega = \begin{cases} 1 & \text{if } \omega = \emptyset \\ 0 & \text{if } \omega \neq \emptyset. \end{cases}$$

One can check that a mould M^\bullet is invertible if and only if M^\emptyset is invertible in \mathbf{A} (see below).

One must in fact regard $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ as an \mathbf{A} -algebra, i.e. its module structure and ring structure are compatible: $\mu \in \mathbf{A} \mapsto \mu 1^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$ is indeed a ring homomorphism, the image of which lies in the center of the ring of moulds. The reader familiar with Bourbaki's *Elements of mathematics* will have recognized in $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ the large algebra (over \mathbf{A}) of the monoid Ω^\bullet (*Alg.*, chap. III, §2, n°10). Other authors use the notation $\mathbf{A}\langle\langle\Omega\rangle\rangle$ or $\mathbf{A}[[T^\Omega]]$ to denote this \mathbf{A} -algebra, viewing it as the completion of the free \mathbf{A} -algebra over Ω for the pseudovaluation ord defined below. The originality of moulds lies in the way they are used:

- the shuffling operation available in the free monoid Ω^\bullet will lead us in Section 5 to single out specific classes of moulds, enjoying certain symmetry or antisymmetry properties of fundamental importance (and this is only a small amount of all the structures used by Écalle in wide-ranging contexts);
- we shall see in Sections 6 and 7 how to contract moulds into “comoulds” (and this yields non-trivial results in the local study of analytic dynamical systems);
- the extra structure of commutative semigroup on Ω will allow us to define another operation, the “composition” of moulds (see below).

There is a pseudovaluation $\text{ord}: \mathcal{M}^\bullet(\Omega, \mathbf{A}) \rightarrow \mathbb{N} \cup \{\infty\}$, which we call “order”: we say that a mould M^\bullet has order $\geq s$ if $M^\omega = 0$ whenever $r(\omega) < s$, and $\text{ord}(M^\bullet)$ is the largest such s . This way, we get a complete pseudovaluation ring $(\mathcal{M}^\bullet(\Omega, \mathbf{A}), \text{ord})$. In fact, if \mathbf{A} is an integral domain (as is the case of $\mathbb{C}[[x]]$), then $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ is an integral domain and ord is a valuation.

4.3 It is easy to construct “mould derivations”, i.e. \mathbb{C} -linear operators D of $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ such that $D(M^\bullet \times N^\bullet) = (DM^\bullet) \times N^\bullet + M^\bullet \times DN^\bullet$.

For instance, for any function $\varphi: \Omega \rightarrow \mathbf{A}$, the formula

$$D_\varphi M^\omega = \begin{cases} 0 & \text{if } \omega = \emptyset \\ (\varphi(\omega_1) + \cdots + \varphi(\omega_r)) M^\omega & \text{if } \omega = (\omega_1, \dots, \omega_r) \end{cases}$$

defines a mould derivation D_φ . With $\varphi \equiv 1$, we get $DM^\omega = r(\omega)M^\omega$.

When Ω is a commutative semigroup (the operation of which is denoted additively), we define the *sum of a non-empty word* as

$$\|\omega\| = \omega_1 + \cdots + \omega_r \in \Omega, \quad \omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet.$$

Then, for any mould U^\bullet such that $U^\emptyset = 0$, the formula

$$\nabla_{U^\bullet} M^\omega = \sum_{\omega=\alpha+\beta+\gamma, \beta \neq \emptyset} U^\beta M^{\alpha \parallel \beta \parallel \gamma} \tag{4.2}$$

defines a mould derivation ∇_{U^\bullet} . The derivation D_φ is nothing but ∇_{U^\bullet} with $U^\omega = \varphi(\omega_1)$ for $\omega = (\omega_1)$ and $U^\omega = 0$ for $r(\omega) \neq 1$.

When $\Omega \subset \mathbf{A}$, an important example is

$$\nabla M^\omega = \|\omega\| M^\omega, \quad (4.3)$$

obtained with $\varphi(\eta) \equiv \eta$. On the other hand, every derivation $d: \mathbf{A} \rightarrow \mathbf{A}$ obviously induces a mould derivation D , the action of which on any mould M^\bullet is defined by

$$DM^\omega = d(M^\omega), \quad \omega \in \Omega^\bullet. \quad (4.4)$$

Remark 4.1. With $\Omega = \mathcal{N}$ defined by (3.2) and $\mathbf{A} = \mathbb{C}[[x]]$, the mould \mathcal{V}^\bullet determined in Lemma 3.2 is the unique solution of the mould equation

$$(D + \nabla)\mathcal{V}^\bullet = J_a^\bullet \times \mathcal{V}^\bullet, \quad (4.5)$$

such that $\mathcal{V}^\emptyset = 1$ and $\mathcal{V}^\omega \in x\mathbb{C}[[x]]$ for $\omega \neq \emptyset$, with D induced by $d = x^2 \frac{d}{dx}$ and

$$J_a^\omega = \begin{cases} a_{\omega_1} & \text{if } \omega = (\omega_1) \\ 0 & \text{if } r(\omega) \neq 1. \end{cases} \quad (4.6)$$

4.4 When Ω is a commutative semigroup, the *composition of moulds* is defined as follows:

$$\begin{aligned} C^\bullet = M^\bullet \circ U^\bullet: \quad \emptyset &\mapsto C^\emptyset = M^\emptyset, \\ \omega \neq \emptyset &\mapsto C^\omega = \sum_{\substack{s \geq 1, \omega^1, \dots, \omega^s \neq \emptyset \\ \omega = \omega^1 \dots \omega^s}} M^{(\|\omega^1\|, \dots, \|\omega^s\|)} U^{\omega^1} \dots U^{\omega^s}, \end{aligned}$$

with summation over all possible decompositions of ω into non-empty words (thus $1 \leq s \leq r(\omega)$ and the sum is finite). The map $M^\bullet \mapsto M^\bullet \circ U^\bullet$ is clearly \mathbf{A} -linear; it is in fact an \mathbf{A} -algebra homomorphism:

$$(M^\bullet \circ U^\bullet) \times (N^\bullet \circ U^\bullet) = (M^\bullet \times N^\bullet) \circ U^\bullet$$

(the verification of this distributivity property is left as an exercise).

Obviously, $1^\bullet \circ U^\bullet = 1^\bullet$ for any mould U^\bullet . The *identity mould*

$$I^\bullet: \omega \mapsto I^\omega = \begin{cases} 1 & \text{if } r(\omega) = 1 \\ 0 & \text{if } r(\omega) \neq 1 \end{cases}$$

satisfies $M^\bullet \circ I^\bullet = M^\bullet$ for any mould M^\bullet . But $I^\bullet \circ U^\bullet = U^\bullet$ only if $U^\emptyset = 0$ (a requirement that we could have imposed when defining mould composition, since the value of U^\emptyset is ignored when computing $M^\bullet \circ U^\bullet$); in general, $I^\bullet \circ U^\bullet = U^\bullet - U^\emptyset 1^\bullet$.

Mould composition is associative⁶ and not commutative. One can check that a mould U^\bullet admits an inverse for composition (a mould V^\bullet such that $V^\bullet \circ U^\bullet =$

⁶Hint: The computation of $M^\bullet \circ (U^\bullet \circ V^\bullet)$ at ω involves all the decompositions $\omega = \omega^1 \dots \omega^s$ into non-empty words and then all the decompositions of each factor ω^i as $\omega^1 = \alpha^1 \dots \alpha^{i_1}, \omega^2 = \alpha^{i_1+1} \dots \alpha^{i_2}, \dots, \omega^s = \alpha^{i_{s-1}+1} \dots \alpha^{i_s}$ (where $1 \leq i_1 < i_2 < \dots < i_s = t$, with each α^j non-

$U^\bullet \circ V^\bullet = I^\bullet$) if and only if U^ω is invertible in \mathbf{A} whenever $r(\omega) = 1$ and $U^\emptyset = 0$. These moulds thus form a group under composition.

In the following, we do not always assume Ω to be a commutative semigroup and mould composition is thus not always defined. However, observe that, in the absence of semigroup structure, the definition of $M^\bullet \circ U^\bullet$ makes sense for any mould M^\bullet such that M^ω only depends on $r(\omega)$ and that most of the above properties can be adapted to this particular situation.

4.5 As an elementary illustration, one can express the multiplicative inverse of a mould M^\bullet with $\mu = M^\emptyset$ invertible as

$$(M^\bullet)^{\times(-1)} = G^\bullet \circ M^\bullet, \quad \text{with } G^\omega = (-1)^{r(\omega)} \mu^{-r(\omega)-1}.$$

Indeed, G^\bullet is nothing but the multiplicative inverse of $\mu 1^\bullet + I^\bullet$ and

$$M^\bullet = \mu 1^\bullet + I^\bullet \circ M^\bullet = (\mu 1^\bullet + I^\bullet) \circ M^\bullet,$$

whence the result follows immediately.

The above computation does not require any semigroup structure on Ω . Besides, one can also write $(M^\bullet)^{\times(-1)} = \sum_{s \geq 0} (-1)^s \mu^{-s-1} (M^\bullet - \mu 1^\bullet)^{\times s}$ (convergent series for the topology of $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ induced by ord).

4.6 We define elementary scalar moulds \exp_t^\bullet , $t \in \mathbb{C}$, and \log^\bullet by the formulas $\exp_t^\omega = \frac{t^{r(\omega)}}{r(\omega)!}$ and

$$\log^\omega = 0 \quad \text{if } \omega = \emptyset, \quad \log^\omega = \frac{(-1)^{r(\omega)-1}}{r(\omega)} \quad \text{if } \omega \neq \emptyset.$$

One can check that

$$\begin{aligned} \exp_0^\bullet &= 1^\bullet, & \exp_{t_1}^\bullet \times \exp_{t_2}^\bullet &= \exp_{t_1+t_2}^\bullet, & t_1, t_2 &\in \mathbb{C}, \\ (\exp_t^\bullet - 1^\bullet) \circ \frac{1}{t} \log^\bullet &= \frac{1}{t} \log^\bullet \circ (\exp_t^\bullet - 1^\bullet) = I^\bullet, & t &\in \mathbb{C}^*, \end{aligned}$$

(use for instance $\exp_t^\bullet = \sum_{s \geq 0} \frac{t^s}{s!} (I^\bullet)^{\times s}$ and $\log^\bullet = \sum_{s \geq 1} \frac{(-1)^{s-1}}{s} (I^\bullet)^{\times s}$; mould composition is well-defined here even if Ω is not a semigroup).

Now, consider on the one hand the Lie algebra

$$\begin{aligned} \mathfrak{L}^\bullet(\Omega, \mathbf{A}) &= \{ U^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A}) \mid U^\emptyset = 0 \}, \\ &\quad \text{with bracketting } [U^\bullet, V^\bullet] = U^\bullet \times V^\bullet - V^\bullet \times U^\bullet, \end{aligned} \tag{4.7}$$

and on the other hand the subgroup

$$G^\bullet(\Omega, \mathbf{A}) = \{ M^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A}) \mid M^\emptyset = 1 \} \tag{4.8}$$

of the multiplicative group of invertible moulds.

Then, for each $U^\bullet \in \mathfrak{L}^\bullet(\Omega, \mathbf{A})$, $(\exp_t^\bullet \circ U^\bullet)_{t \in \mathbb{C}}$ is a one-parameter group inside $G^\bullet(\Omega, \mathbf{A})$. Moreover, the map

$$E_t : U^\bullet \in \mathfrak{L}^\bullet(\Omega, \mathbf{A}) \mapsto M^\bullet = \exp_t^\bullet \circ U^\bullet \in G^\bullet(\Omega, \mathbf{A})$$

empty); it is equivalent to sum first over all the decompositions $\omega = \alpha^1 \dots \alpha^t$ and then to consider all manners of regrouping adjacent factors $(\alpha^1 \dots \alpha^{i_1}) \cdot (\alpha^{i_1+1} \dots \alpha^{i_2}) \dots (\alpha^{i_{s-1}+1} \dots \alpha^{i_s})$, which yields the value of $(M^\bullet \circ U^\bullet) \circ V^\bullet$ at ω .

is a bijection for each $t \in \mathbb{C}^*$ (with reciprocal $M^\bullet \mapsto \frac{1}{t} \log^\bullet \circ U^\bullet$), which allows us to consider $\mathcal{L}^\bullet(\Omega, \mathbf{A})$ as the Lie algebra of $G^\bullet(\Omega, \mathbf{A})$ in the sense that

$$[U^\bullet, V^\bullet] = \frac{d}{dt} \left(E_t(U^\bullet) \times V^\bullet \times E_t(U^\bullet)^{\times(-1)} \right)_{|t=0}.$$

Observe that mould composition is not necessary to define the map E_t and its reciprocal: one can use the series

$$E_t(U^\bullet) = \sum_{s \geq 0} \frac{t^s}{s!} (U^\bullet)^{\times s}, \quad E_t^{-1}(M^\bullet) = \frac{1}{t} \sum_{s \geq 1} \frac{(-1)^{s-1}}{s} (M^\bullet - 1^\bullet)^{\times s} \quad (4.9)$$

(they are formally convergent because $\text{ord}(U^\bullet)$ and $\text{ord}(M^\bullet - 1^\bullet) \geq 1$).

5 Alternality and symmetrality

5.1 Even if Ω is not a semigroup, another operation available in Ω^\bullet is *shuffling*: if two non-empty words $\omega^1 = (\omega_1, \dots, \omega_\ell)$ and $\omega^2 = (\omega_{\ell+1}, \dots, \omega_r)$ are given, one says that a word ω belongs to their shuffling if it can be written $(\omega_{\sigma(1)}, \dots, \omega_{\sigma(r)})$ with a permutation σ such that $\sigma(1) < \dots < \sigma(\ell)$ and $\sigma(\ell+1) < \dots < \sigma(r)$ (in other words, ω can be obtained by interdigitating the letters of ω^1 and those of ω^2 while preserving their internal order in ω^1 or ω^2). We denote by $\text{sh} \binom{\omega^1, \omega^2}{\omega}$ the number of such permutations σ , and we set $\text{sh} \binom{\omega^1, \omega^2}{\omega} = 0$ if ω does not belong to the shuffling of ω^1 and ω^2 .

Definition 5.1. A mould M^\bullet is said to be alternal if $M^\emptyset = 0$ and, for any two non-empty words ω^1, ω^2 ,

$$\sum_{\omega \in \Omega^\bullet} \text{sh} \binom{\omega^1, \omega^2}{\omega} M^\omega = 0. \quad (5.1)$$

It is said to be symmetral if $M^\emptyset = 1$ and, for any two non-empty words ω^1, ω^2 ,

$$\sum_{\omega \in \Omega^\bullet} \text{sh} \binom{\omega^1, \omega^2}{\omega} M^\omega = M^{\omega^1} M^{\omega^2}. \quad (5.2)$$

Of course the above sums always have finite support. For instance, if $\omega^1 = (\omega_1)$ and $\omega^2 = (\omega_2, \omega_3)$, the left-hand side in both previous formulas is $M^{\omega_1, \omega_2, \omega_3} + M^{\omega_2, \omega_1, \omega_3} + M^{\omega_2, \omega_3, \omega_1}$.

The motivation for this definition lies in formula (7.2) below. We shall see in Section 7 the interpretation of alternality or symmetrality in terms of the operators obtained by mould-comould expansions: alternal moulds will be related to the Lie algebra of derivations, symmetral moulds to the group of automorphisms.

Alternal (resp. symmetral) moulds have to do with primitive (resp. group-like) elements of a certain graded cocommutative Hopf algebra, at least when \mathbf{A} is a field—see the remark on Lemma 5.3 below.

An obvious example of alternal mould is I^\bullet , or any mould J^\bullet such that $J^\omega = 0$ for $r(\omega) \neq 1$ (as is the case of J_a^\bullet defined by (4.6)). An elementary example of symmetral mould is \exp_t^\bullet for any $t \in \mathbb{C}$; a non-trivial example is the mould \mathcal{V}^\bullet determined by Lemma 3.2, the symmetrality of which is the object of Proposition 5.5 below. The mould \log^\bullet is not alternal (nor symmetral), but “alternel”; alternelity and symmetrelity are two other types of symmetry introduced by Écalle, parallel to altermality and symmetrality, but we shall not be concerned with them in this text (see however the end of Section 7).

The next paragraphs contain the proof of the following properties:

Proposition 5.1. *Alternal moulds form a Lie subalgebra $\mathfrak{L}_{alt}^\bullet(\Omega, \mathbf{A})$ of the Lie algebra $\mathfrak{L}^\bullet(\Omega, \mathbf{A})$ defined by (4.7). Symmetral moulds form a subgroup $G_{sym}^\bullet(\Omega, \mathbf{A})$ of the multiplicative group $G^\bullet(\Omega, \mathbf{A})$ defined by (4.8). The map E_t defined by (4.9) induces a bijection from $\mathfrak{L}_{alt}^\bullet(\Omega, \mathbf{A})$ to $G_{sym}^\bullet(\Omega, \mathbf{A})$ for each $t \in \mathbb{C}^*$.*

Proposition 5.2. *Given a mould M^\bullet , we define a mould $SM^\bullet = \widetilde{M}^\bullet$ by the formulas*

$$\widetilde{M}^\emptyset = M^\emptyset, \quad \widetilde{M}^{\omega_1, \dots, \omega_r} = (-1)^r M^{\omega_r, \dots, \omega_1}, \quad r \geq 1, \omega_1, \dots, \omega_r \in \Omega. \quad (5.3)$$

Then S is an involution and an antihomomorphism of the \mathbf{A} -algebra $\mathcal{M}^\bullet(\Omega, \mathbf{A})$, and

$$\begin{aligned} M^\bullet \text{ alternal} &\Rightarrow SM^\bullet = -M^\bullet, \\ M^\bullet \text{ symmetral} &\Rightarrow SM^\bullet = (M^\bullet)^{\times(-1)} \text{ (multiplicative inverse).} \end{aligned}$$

Proposition 5.3. *If Ω is a commutative semigroup and U^\bullet is alternal, then*

$$M^\bullet \text{ alternal} \Rightarrow M^\bullet \circ U^\bullet \text{ alternal,} \quad (5.4)$$

$$M^\bullet \text{ symmetral} \Rightarrow M^\bullet \circ U^\bullet \text{ symmetral.} \quad (5.5)$$

If moreover U^\bullet admits an inverse for composition (i.e. if U^ω has a multiplicative inverse in \mathbf{A} whenever $r(\omega) = 1$), then this inverse is alternal itself; thus alternal invertible moulds form a subgroup of the group (for composition) of invertible moulds.

Proposition 5.4. *If D is a mould derivation induced by a derivation of \mathbf{A} , or of the form D_φ with $\varphi: \Omega \rightarrow \mathbf{A}$, or of the form ∇_{J^\bullet} with J^\bullet alternal (with the assumption that Ω is a commutative semigroup in this last case), and if M^\bullet is symmetral, then $(DM^\bullet) \times (M^\bullet)^{\times(-1)}$ and $(M^\bullet)^{\times(-1)} \times (DM^\bullet)$ are alternal.*

5.2 The following definition will facilitate the proof of most of these properties and enlighten the connection with derivations and algebra automorphisms to be discussed in Section 7.

Definition 5.2. We call dimould⁷ any map $\mathbf{M}^{\bullet,\bullet}$ from $\Omega^\bullet \times \Omega^\bullet$ to \mathbf{A} ; its value on (ω, η) is denoted $\mathbf{M}^{\omega,\eta}$. The set of dimoulds is denoted $\mathcal{M}^{\bullet,\bullet}(\Omega, \mathbf{A})$; when viewed as the large algebra of the monoid $\Omega^\bullet \times \Omega^\bullet$, it is a non-commutative \mathbf{A} -algebra.

Observe that, the monoid law on $\Omega^\bullet \times \Omega^\bullet$ being

$$\varpi^1 = (\omega^1, \eta^1), \quad \varpi^2 = (\omega^2, \eta^2) \Rightarrow \varpi^1 \cdot \varpi^2 = (\omega^1 \cdot \eta^1, \omega^2 \cdot \eta^2),$$

the finitess of the number of decompositions of any $\varpi \in \Omega^\bullet \times \Omega^\bullet$ as $\varpi = \varpi^1 \cdot \varpi^2$ allows us to consider this large algebra, in which the multiplication is defined by a formula similar to (4.1). The unit of dimould multiplication is $1^{\bullet,\bullet}: (\omega, \eta) \mapsto 1$ if $\omega = \eta = \emptyset$ and 0 otherwise.

Lemma 5.1. *The map $\tau: M^\bullet \in \mathcal{M}^{\bullet}(\Omega, \mathbf{A}) \mapsto \mathbf{M}^{\bullet,\bullet} \in \mathcal{M}^{\bullet,\bullet}(\Omega, \mathbf{A})$ defined by*

$$\mathbf{M}^{\alpha,\beta} = \sum_{\omega \in \Omega^\bullet} \text{sh} \left(\frac{\alpha, \beta}{\omega} \right) M^\omega, \quad \alpha, \beta \in \Omega^\bullet$$

is an \mathbf{A} -algebra homomorphism.

Proof. The map τ is clearly \mathbf{A} -linear and $\tau(1^\bullet) = 1^{\bullet,\bullet}$. Let $P^\bullet = M^\bullet \times N^\bullet$ and $\mathbf{P}^{\bullet,\bullet} = \tau(P^\bullet)$; since

$$\mathbf{P}^{\alpha,\beta} = \sum_{\gamma^1, \gamma^2 \in \Omega^\bullet} \text{sh} \left(\frac{\alpha, \beta}{\gamma^1 \cdot \gamma^2} \right) M^{\gamma^1} N^{\gamma^2}, \quad \alpha, \beta \in \Omega^\bullet,$$

the property $\mathbf{P}^{\bullet,\bullet} = \tau(M^\bullet) \times \tau(N^\bullet)$ follows from the identity

$$\text{sh} \left(\frac{\alpha, \beta}{\gamma^1 \cdot \gamma^2} \right) = \sum_{\alpha=\alpha^1 \cdot \alpha^2, \beta=\beta^1 \cdot \beta^2} \text{sh} \left(\frac{\alpha^1, \beta^1}{\gamma^1} \right) \text{sh} \left(\frac{\alpha^2, \beta^2}{\gamma^2} \right) \quad (5.6)$$

(the verification of which is left to the reader). \square

As in the case of moulds, we can define the “order” of a dimould and get a pseudovaluation $\text{ord}: \mathcal{M}^{\bullet,\bullet}(\Omega, \mathbf{A}) \rightarrow \mathbb{N} \cup \{\infty\}$: by definition $\text{ord}(\mathbf{M}^{\bullet,\bullet}) \geq s$ if $\mathbf{M}^{\omega,\eta} = 0$ whenever $r(\omega) + r(\eta) < s$. We then get a complete pseudovaluation ring $(\mathcal{M}^{\bullet,\bullet}(\Omega, \mathbf{A}), \text{ord})$ and the homomorphism τ is continuous since $\text{ord}(\tau(M^\bullet)) \geq \text{ord}(M^\bullet)$.

Definition 5.3. We call decomposable a dimould $\mathbf{P}^{\bullet,\bullet}$ of the form $\mathbf{P}^{\omega,\eta} = M^\omega N^\eta$ (for all $\omega, \eta \in \Omega^\bullet$), where M^\bullet and N^\bullet are two moulds. We then use the notation $\mathbf{P}^{\bullet,\bullet} = M^\bullet \otimes N^\bullet$.

One can check that the relation

$$(M_1^\bullet \otimes N_1^\bullet) \times (M_2^\bullet \otimes N_2^\bullet) = (M_1^\bullet \times M_2^\bullet) \otimes (M_2^\bullet \times N_2^\bullet) \quad (5.7)$$

⁷Not to be confused with the *bimoulds* introduced by Écalle in connection with Multizeta values, which correspond to the case where the set Ω itself is the cartesian product of two sets—see the end of Section 13.

holds in $\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$, for any four moulds $M_1^\bullet, N_1^\bullet, M_2^\bullet, N_2^\bullet$.

With this notation for decomposable dimoulds, we can now rephrase Definition 5.1 with the help of the homomorphism τ of Lemma 5.1:

Lemma 5.2. *A mould M^\bullet is alternal iff $\tau(M^\bullet) = M^\bullet \otimes 1^\bullet + 1^\bullet \otimes M^\bullet$. A mould M^\bullet is symmetral iff $M^\emptyset = 1$ and $\tau(M^\bullet) = M^\bullet \otimes M^\bullet$.*

Notice that the image of τ is contained in the set of *symmetric dimoulds*, i.e. those $\mathbf{M}^{\bullet,\bullet}$ such that $\mathbf{M}^{\alpha,\beta} = \mathbf{M}^{\beta,\alpha}$, because of the obvious relation

$$\text{sh} \begin{pmatrix} \alpha, \beta \\ \omega \end{pmatrix} = \text{sh} \begin{pmatrix} \beta, \alpha \\ \omega \end{pmatrix}, \quad \alpha, \beta, \omega \in \Omega^\bullet. \quad (5.8)$$

5.3 Remark on Definition 5.3. The tensor product is used here as a mere notation, which is related to the tensor product of \mathbf{A} -algebras as follows: there is a unique \mathbf{A} -linear map $\rho: \mathcal{M}^\bullet(\Omega, \mathbf{A}) \otimes_{\mathbf{A}} \mathcal{M}^\bullet(\Omega, \mathbf{A}) \rightarrow \mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$ such that $\rho(M^\bullet \otimes N^\bullet)$ is the above dimould $\mathbf{P}^{\bullet,\bullet}$. The map ρ is an \mathbf{A} -algebra homomorphism, according to (5.7), however its injectivity is not obvious when \mathbf{A} is not a field, and denoting $\rho(M^\bullet \otimes N^\bullet)$ simply as $M^\bullet \otimes N^\bullet$, as in Definition 5.3, is thus an abuse of notation.

In fact, if \mathbf{A} is an integral domain, then the \mathbf{A} -module $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ is torsion-free ($\mu M^\bullet = 0$ implies $\mu = 0$ or $M^\bullet = 0$) and $\text{Ker } \rho$ coincides with the set \mathcal{T} of all torsion elements of $\mathcal{M}^\bullet(\Omega, \mathbf{A}) \otimes_{\mathbf{A}} \mathcal{M}^\bullet(\Omega, \mathbf{A})$. Indeed, for any $\xi \in \mathcal{T}$, there is a non-zero $\mu \in \mathbf{A}$ such that $\mu\xi = 0$, thus $\mu\rho(\xi) = 0$ in $\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$, whence $\rho(\xi) = 0$. Conversely, suppose $\xi = \sum_{i=1}^n M_i^\bullet \otimes N_i^\bullet \in \text{Ker } \rho$, where the moulds M_i^\bullet are not all zero; without loss of generality we can suppose $M_n^\bullet \neq 0$ and choose $\omega^1 \in \Omega^\bullet$ such that $\mu_n = M_n^{\omega^1} \neq 0$. Setting $\mu_i = M_i^{\omega^1}$ for the other i 's, we get $\sum_{i=0}^n \mu_i N_i^\bullet = 0$, whence $\mu_n \xi = \sum_{i=0}^{n-1} (\mu_n M_i^\bullet - \mu_i M_n^\bullet) \otimes N_i^\bullet$, still with $\mu_n \xi \in \text{Ker } \rho$. By induction on n , one gets a non-zero $\mu \in \mathbf{A}$ such that $\mu\xi = 0$.

Therefore, ρ is injective when \mathbf{A} is a principal integral domain, as is the case of $\mathbb{C}[[x]]$, because any torsion-free \mathbf{A} -module is then flat (Bourbaki, *Alg. comm.*, chap. I, §2, n°4, Prop. 3), hence its tensor product with itself is also torsion-free (by flatness, the injectivity of $\phi: M^\bullet \mapsto \mu M^\bullet$, for $\mu \neq 0$, implies the injectivity of $\phi \otimes \text{Id}: \xi \mapsto \mu\xi$).

This is a fortiori the case when \mathbf{A} is a field; this is used in the remark on Lemma 5.3 below.

5.4 Proof of Proposition 5.1. The set $\mathcal{L}_{\text{alt}}^\bullet(\Omega, \mathbf{A})$ of alternal moulds is clearly an \mathbf{A} -submodule of $\mathcal{L}^\bullet(\Omega, \mathbf{A})$. Given U^\bullet and V^\bullet in this set, the altermality of $[U^\bullet, V^\bullet]$ is easily checked with the help of Lemma 5.1, formula (5.7) and Lemma 5.2.

Let M^\bullet and N^\bullet be symmetral. The symmetrality of $M^\bullet \times N^\bullet$ follows from Lemma 5.1, formula (5.7) and Lemma 5.2. Similarly, the multiplicative inverse \widetilde{M}^\bullet of M^\bullet satisfies $\tau(\widetilde{M}^\bullet) \times \tau(M^\bullet) = \tau(M^\bullet) \times \tau(\widetilde{M}^\bullet) = \tau(1^\bullet) = 1^{\bullet,\bullet}$, by uniqueness of the multiplicative inverse in $\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$ it follows that $\tau(\widetilde{M}^\bullet) = \widetilde{M}^\bullet \otimes \widetilde{M}^\bullet$ and \widetilde{M}^\bullet is symmetral.

Now let $t \in \mathbb{C}^*$. Suppose first $U^\bullet \in \mathcal{L}_{\text{alt}}^\bullet(\Omega, \mathbf{A})$. We check that $M^\bullet = E_t(U^\bullet)$ is symmetral by using the continuity of τ and formula (4.9): $\tau(M^\bullet) = \exp(a+b)$ with $a = tU^\bullet \otimes 1^\bullet$ and $b = 1^\bullet \otimes tU^\bullet$, where the exponential series is well-defined in

$\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$ because $\text{ord}(a), \text{ord}(b) > 0$; since $a \times b = b \times a$, the standard properties of the exponential series yield

$$\tau(M^\bullet) = \exp(a) \times \exp(b) = (\exp(tU^\bullet) \otimes 1^\bullet) \times (1^\bullet \otimes \exp(tU^\bullet)) = M^\bullet \otimes M^\bullet.$$

Conversely, supposing $M^\bullet = 1^\bullet + N^\bullet \in G_{\text{sym}}(\Omega, \mathbf{A})$, we check that $U^\bullet = E_t^{-1}(M^\bullet)$ is alternal: by continuity, we can apply τ termwise to the logarithm series in (4.9) and write $\tau(M^\bullet - 1^\bullet) = N^\bullet \otimes 1^\bullet + 1^\bullet \otimes N^\bullet + N^\bullet \otimes N^\bullet = a + b + a \times b$, with $a = N^\bullet \otimes 1^\bullet$ and $b = 1^\bullet \otimes N^\bullet$ commuting in $\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$, the conclusion then follows from the identity

$$\sum_{s \geq 1} \frac{(-1)^{s-1}}{s} (a + b + a \times b)^{\times s} = \sum_{s \geq 1} \frac{(-1)^{s-1}}{s} a^{\times s} + \sum_{s \geq 1} \frac{(-1)^{s-1}}{s} b^{\times s}$$

(which follows from the observation that, given $c \in \mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$ with $\text{ord}(c) > 0$, $\sum \frac{(-1)^{s-1}}{s} c^{\times s}$ is the only dimould ℓ of positive order such that $\exp(\ell) = 1^{\bullet,\bullet} + c$).

5.5 Proof of Proposition 5.2. It is obvious that S is an involution and the identity

$$S(M^\bullet \times N^\bullet) = S N^\bullet \times S M^\bullet, \quad M^\bullet, N^\bullet \in \mathcal{M}^{\bullet}(\Omega, \mathbf{A})$$

clearly follows from the Definition (4.1) of mould multiplication. Let us define an \mathbf{A} -linear map

$$\xi : \mathbf{M}^{\bullet,\bullet} \in \mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A}) \mapsto P^\bullet = \xi(\mathbf{M}^{\bullet,\bullet}) \in \mathcal{M}^{\bullet}(\Omega, \mathbf{A})$$

by the formula

$$P^\omega = \sum_{\omega=\alpha+\beta} (-1)^{r(\alpha)} \mathbf{M}^{\tilde{\alpha}, \beta}, \quad \omega \in \Omega^\bullet, \quad (5.9)$$

where $\tilde{\alpha} = (\omega_i, \dots, \omega_1)$ for $\alpha = (\omega_1, \dots, \omega_i)$ with $i \geq 1$ and $\tilde{\emptyset} = \emptyset$. Thus

$$\begin{aligned} P^\emptyset &= \mathbf{M}^{\emptyset, \emptyset}, & P^{(\omega_1)} &= \mathbf{M}^{\emptyset, (\omega_1)} - \mathbf{M}^{(\omega_1), \emptyset}, \\ P^{(\omega_1, \omega_2)} &= \mathbf{M}^{\emptyset, (\omega_1, \omega_2)} - \mathbf{M}^{(\omega_1), (\omega_2)} + \mathbf{M}^{(\omega_2, \omega_1), \emptyset}, \end{aligned}$$

and so on. The rest of Proposition 5.2 follows from

Lemma 5.3. *For any two moulds M^\bullet, N^\bullet , one has*

$$\xi(M^\bullet \otimes N^\bullet) = (S M^\bullet) \times N^\bullet, \quad (5.10)$$

$$\xi \circ \tau(M^\bullet) = M^\emptyset 1^\bullet, \quad (5.11)$$

with the homomorphism τ of Lemma 5.1.

Indeed, if M^\bullet is alternal, then

$$S M^\bullet + M^\bullet = (S M^\bullet) \times 1^\bullet + 1^\bullet \times M^\bullet = \xi(M^\bullet \otimes 1^\bullet + 1^\bullet \otimes M^\bullet) = \xi \circ \tau(M^\bullet) = 0,$$

and if M^\bullet is symmetral, then

$$(S M^\bullet) \times M^\bullet = \xi(M^\bullet \otimes M^\bullet) = \xi \circ \tau(M^\bullet) = 1^\bullet$$

and similarly $M^\bullet \times S M^\bullet = 1^\bullet$ because $S M^\bullet$ is clearly symmetral too.

Proof of Lemma 5.3. Formula (5.10) is obvious. Let $\mathbf{M}^{\bullet,\bullet} = \tau(M^\bullet)$ and $P^\bullet = \xi(\mathbf{M}^{\bullet,\bullet})$. Clearly $P^\emptyset = \mathbf{M}^{\emptyset,\emptyset} = M^\emptyset$. Let $\omega = (\omega_1, \dots, \omega_r)$ with $r \geq 1$: we must show that $P^\omega = 0$.

Using the notations $\alpha^i = (\omega_1, \dots, \omega_i)$ and $\beta^i = (\omega_{i+1}, \dots, \omega_r)$ for $0 \leq i \leq r$ (with $\alpha^0 = \beta^r = \emptyset$), we can write $P^\omega = \sum_{i=0}^r (-1)^i \mathbf{M}^{\tilde{\alpha}^i, \beta^i}$; we then split the sum

$$\mathbf{M}^{\tilde{\alpha}^i, \beta^i} = \sum_{\gamma} \text{sh}(\tilde{\alpha}^i, \beta^i) M^\gamma$$

according to the first letter of the mute variable: $\mathbf{M}^{\tilde{\alpha}^i, \beta^i} = Q_i + R_i$ with

$$\begin{aligned} Q_i &= \sum_{\gamma} \text{sh}(\tilde{\alpha}^{i-1}, \beta^i) M^{(\omega_i)\bullet\gamma} \quad \text{if } 1 \leq i \leq r, & Q_0 &= 0, \\ R_i &= \sum_{\gamma} \text{sh}(\tilde{\alpha}^i, \beta^{i+1}) M^{(\omega_{i+1})\bullet\gamma} \quad \text{if } 1 \leq i \leq r-1, & R_r &= 0. \end{aligned}$$

But, if $0 \leq i \leq r-1$, $Q_{i+1} = R_i$, whence $P^\omega = \sum_{i=1}^r (-1)^i Q_i + \sum_{i=0}^{r-1} (-1)^i Q_{i+1} = 0$.

5.6 Remark on Lemma 5.3. Although this will not be used in the rest of the article, it is worth noting here that the structure we have on $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ is very reminiscent of that of a *cocommutative Hopf algebra*: the algebra structure is given by mould multiplication (4.1), with its unit 1^\bullet ; as for the cocommutative cogebraby structure, we may think of the map $\varepsilon: M^\bullet \mapsto M^\emptyset$ as of a *counit* and of the homomorphism τ as of a kind of *coproduct* (although its range is not exactly $\mathcal{M}^\bullet(\Omega, \mathbf{A}) \otimes_{\mathbf{A}} \mathcal{M}^\bullet(\Omega, \mathbf{A})$); we now may consider that the involution $S: M^\bullet \mapsto \widetilde{M}^\bullet$ behaves as an *antipode*.

Indeed, the identity⁸ $\tau(M^\bullet)^{\emptyset, \alpha} = \tau(M^\bullet)^{\alpha, \emptyset} = M^\alpha$ can be interpreted as a counit-like property for ε and the fact that any dimould in the image of ρ is symmetric (consequence of (5.8)) as a cocommutativity-like property, in the sense that $\tau(M^\bullet) = \sum P_i^\bullet \otimes Q_i^\bullet$ implies $\sum \varepsilon(P_i^\bullet) Q_i^\bullet = \sum \varepsilon(Q_i^\bullet) P_i^\bullet = M^\bullet$ and $\sum P_i^\bullet \otimes Q_i^\bullet = \sum Q_i^\bullet \otimes P_i^\bullet$. The analogue of coassociativity for τ is obtained by considering the maps τ_ℓ and τ_r which associate with any dimould $\mathbf{M}^{\bullet,\bullet}$ the “trimoulds” $\mathbf{P}^{\bullet,\bullet,\bullet} = \tau_\ell(\mathbf{M}^{\bullet,\bullet})$ and $\mathbf{Q}^{\bullet,\bullet,\bullet} = \tau_r(\mathbf{M}^{\bullet,\bullet})$ defined by

$$\mathbf{P}^{\alpha, \beta, \gamma} = \sum_{\eta \in \Omega^\bullet} \text{sh}(\alpha, \beta) \mathbf{M}^{\eta, \gamma}, \quad \mathbf{Q}^{\alpha, \beta, \gamma} = \sum_{\eta \in \Omega^\bullet} \text{sh}(\beta, \gamma) \mathbf{M}^{\alpha, \eta}$$

and by observing⁹ that $\tau_\ell \circ \tau = \tau_r \circ \tau$: when $\tau(M^\bullet) = \sum_i P_i^\bullet \otimes Q_i^\bullet$ with $\tau(P_i^\bullet) = \sum_j A_{i,j}^\bullet \otimes B_{i,j}^\bullet$ and $\tau(Q_i^\bullet) = \sum_k C_{i,k}^\bullet \otimes D_{i,k}^\bullet$, this yields

$$\sum_{i,j} A_{i,j}^\bullet \otimes B_{i,j}^\bullet \otimes Q_i^\bullet = \sum_{i,k} P_i^\bullet \otimes C_{i,k}^\bullet \otimes D_{i,k}^\bullet.$$

⁸derived from the obvious relation $\text{sh}(\emptyset, \alpha) = \text{sh}(\alpha, \emptyset) = 1_{\{\omega=\alpha\}}$.

⁹Proof: for a mould M^\bullet , we have $\tau_\ell \circ \tau(M^\bullet)^{\alpha, \beta, \gamma} = \sum_{\omega} \text{sh}(\alpha, \beta, \gamma) M^\omega$ with $\text{sh}(\alpha, \beta, \gamma) = \sum_{\eta} \text{sh}(\alpha, \beta) \text{sh}(\eta, \gamma)$ coinciding with $\sum_{\eta} \text{sh}(\alpha, \eta) \text{sh}(\beta, \gamma)$, hence $\tau_r \circ \tau(M^\bullet)^{\alpha, \beta, \gamma} = \sum_{\omega} \text{sh}(\alpha, \beta, \gamma) M^\omega$ as well.

Finally, the compatibility of ε , τ and S is expressed through formulas (5.10)–(5.11) (complemented by relations $\xi'(M^\bullet \otimes N^\bullet) = M^\bullet \times SN^\bullet$ and $\xi' \circ \tau(M^\bullet) = M^\emptyset 1^\bullet$ involving a map ξ' defined by replacing $(-1)^{r(\alpha)} M^{\tilde{\alpha}, \beta}$ with $(-1)^{r(\beta)} M^{\alpha, \tilde{\beta}}$ in (5.9)); therefore

$$\tau(M^\bullet) = \sum_i P_i^\bullet \otimes Q_i^\bullet \Rightarrow \sum_i SP_i^\bullet \times Q_i^\bullet = M^\emptyset 1^\bullet = \sum_i P_i^\bullet \times SQ_i^\bullet.$$

When \mathbf{A} is a field, we get a true cocommutative Hopf algebra (graded by ord) by considering $\mathcal{H}^\bullet(\Omega, \mathbf{A}) = \tau^{-1}(\mathcal{B})$ with $\mathcal{B} = \mathcal{M}^\bullet(\Omega, \mathbf{A}) \otimes_{\mathbf{A}} \mathcal{M}^\bullet(\Omega, \mathbf{A})$ (we can view \mathcal{B} as a subalgebra of $\mathcal{M}^{\bullet\bullet}(\Omega, \mathbf{A})$ according to the remark on Definition 5.3). Indeed, in view of the above, it suffices essentially to check that $M^\bullet \in \mathcal{H} = \mathcal{H}^\bullet(\Omega, \mathbf{A})$ implies $\tau(M^\bullet) \in \mathcal{H} \otimes_{\mathbf{A}} \mathcal{H}$ (and not only $\tau(M^\bullet) \in \mathcal{B}$), so that the restriction of the homomorphism τ to \mathcal{H} is a *bona fide* coproduct

$$\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes_{\mathbf{A}} \mathcal{H}.$$

This can be done by choosing a minimal N such that $\tau(M^\bullet)$ can be written as a sum of N decomposable dimoulds: $\tau(M^\bullet) = \sum_{i=1}^N P_i^\bullet \otimes Q_i^\bullet$ then implies that the Q_i^\bullet 's are linearly independent over \mathbf{A} and the coassociativity property allows one to show that each P_i^\bullet lies in \mathcal{H} (choose a basis of $\mathcal{M}^\bullet(\Omega, \mathbf{A})$, the first N vectors of which are $Q_1^\bullet, \dots, Q_N^\bullet$, and call ξ_1, \dots, ξ_N the first N covectors of the dual basis: the coassociativity identity can be written $\sum_i \tau(P_i^\bullet)^{\alpha, \beta} Q_i^\gamma = \sum_j P_i^\alpha \tau(Q_j^\bullet)^{\beta, \gamma}$, thus $\tau(P_i^\bullet) = \sum_j P_j^\bullet \otimes N_{i,j}^\beta$ with $N_{i,j}^\beta = \xi_i(\tau(Q_j^\bullet)^{\beta, \bullet})$, hence $P_i^\bullet \in \mathcal{H}$); similarly each Q_i^\bullet lies in \mathcal{H} .

By definition, all the alternal and symmetral moulds belong to this Hopf algebra \mathcal{H} , in which they appear respectively as *primitive* and *group-like* elements.

Finally, when \mathbf{A} is only supposed to be an integral domain, $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ can be viewed as a subalgebra of $\mathcal{M}^\bullet(\Omega, K)$, where K denotes the fraction field of \mathbf{A} ; the \mathbf{A} -valued alternal and symmetral moulds belong to the corresponding Hopf algebra $\mathcal{H}^\bullet(\Omega, K)$.

5.7 Proof of Proposition 5.3. The structure of commutative semigroup on Ω allows us to define a composition involving a dimould and a mould as follows: $\mathbf{C}^{\bullet, \bullet} = \mathbf{M}^{\bullet, \bullet} \circ U^\bullet$ if, for all $\alpha, \beta \in \Omega^\bullet$,

$$\mathbf{C}^{\alpha, \beta} = \sum \mathbf{M}^{(\|\alpha^1\|, \dots, \|\alpha^s\|), (\|\beta^1\|, \dots, \|\beta^t\|)} U^{\alpha^1} \dots U^{\alpha^s} U^{\beta^1} \dots U^{\beta^t},$$

with summation over all possible decompositions of α and β into non-empty words; when α is the empty word, the convention is to replace $(\|\alpha^1\|, \dots, \|\alpha^s\|)$ by \emptyset and $U^{\alpha^1} \dots U^{\alpha^s}$ by 1, and similarly when β is the empty word.

One can check that $\mathbf{M}^{\bullet, \bullet} \circ I^\bullet = \mathbf{M}^{\bullet, \bullet}$ and $\mathbf{M}^{\bullet, \bullet} \circ (U^\bullet \circ V^\bullet) = (\mathbf{M}^{\bullet, \bullet} \circ U^\bullet) \circ V^\bullet$ for any dimould $\mathbf{M}^{\bullet, \bullet}$ and any two moulds U^\bullet, V^\bullet (by the same argument as for the associativity of mould composition).

Proposition 5.3 will follow from

Lemma 5.4. *For any three moulds $M^\bullet, N^\bullet, U^\bullet$,*

$$(M^\bullet \otimes N^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes (N^\bullet \circ U^\bullet). \quad (5.12)$$

For any two moulds M^\bullet, U^\bullet ,

$$U^\bullet \text{ alternal} \Rightarrow \tau(M^\bullet \circ U^\bullet) = \tau(M^\bullet) \circ U^\bullet. \quad (5.13)$$

Proof. The identity (5.12) is an easy consequence of the definition of mould composition in Section 4. As for (5.13), let us suppose U^\bullet alternal and let $\mathbf{M}^{\bullet,\bullet} = \tau(M^\bullet)$, $\mathbf{U}^{\bullet,\bullet} = \tau(U^\bullet)$, $\mathbf{C}^{\bullet,\bullet} = \tau(M^\bullet \circ U^\bullet)$. We have $\mathbf{C}^{\emptyset,\emptyset} = M^\emptyset = \mathbf{M}^{\emptyset,\emptyset}$, as desired. Suppose now α or $\beta \neq \emptyset$, then

$$\mathbf{C}^{\alpha,\beta} = \sum_{s \geq 1, \gamma^1, \dots, \gamma^s \neq \emptyset} \operatorname{sh} \begin{pmatrix} \alpha, \beta \\ \gamma^1 \dots \gamma^s \end{pmatrix} M^{(\|\gamma^1\|, \dots, \|\gamma^s\|)} U^{\gamma^1} \dots U^{\gamma^s}.$$

Using the identity (which is an easy generalisation of (5.6))

$$\operatorname{sh} \begin{pmatrix} \alpha, \beta \\ \gamma^1 \dots \gamma^s \end{pmatrix} = \sum_{\alpha=\alpha^1 \dots \alpha^s, \beta=\beta^1 \dots \beta^s} \operatorname{sh} \begin{pmatrix} \alpha^1, \beta^1 \\ \gamma^1 \end{pmatrix} \dots \operatorname{sh} \begin{pmatrix} \alpha^s, \beta^s \\ \gamma^s \end{pmatrix}, \quad (5.14)$$

with possibly empty factors α^i, β^i , we get

$$\mathbf{C}^{\alpha,\beta} = \sum_{s \geq 1, \alpha=\alpha^1 \dots \alpha^s, \beta=\beta^1 \dots \beta^s} M^{(\|\alpha^1\| + \|\beta^1\|, \dots, \|\alpha^s\| + \|\beta^s\|)} \mathbf{U}^{\alpha^1, \beta^1} \dots \mathbf{U}^{\alpha^s, \beta^s}, \quad (5.15)$$

with the convention $\|\emptyset\| = 0$. Observe that this last summation involves only finitely many nonzero terms because $\alpha^i = \beta^i = \emptyset$ implies $\mathbf{U}^{\alpha^i, \beta^i} = 0$.

If α or β is the empty word, since $\mathbf{U}^{\omega, \emptyset} = \mathbf{U}^{\emptyset, \omega} = U^\omega$ we obtain that the values of $\mathbf{C}^{\bullet,\bullet}$ and $\mathbf{M}^{\bullet,\bullet} \circ U^\bullet$ at (α, β) coincide. If neither α nor β is empty, then we have moreover $\mathbf{U}^{\alpha^i, \beta^i} \neq 0 \Rightarrow \alpha^i$ or $\beta^i = \emptyset$, thus (5.15) can be rewritten (retaining only non-empty factors)

$$\mathbf{C}^{\alpha,\beta} = \sum_{\omega \in \Omega^\bullet} \sum_{\alpha=\alpha^1 \dots \alpha^s, \beta=\beta^1 \dots \beta^t} \operatorname{sh} \left((\|\alpha^1\|, \dots, \|\alpha^s\|), (\|\beta^1\|, \dots, \|\beta^t\|) \right) M^\omega U^{\alpha^1} \dots U^{\alpha^s} U^{\beta^1} \dots U^{\beta^t},$$

with the first summation over all possible decompositions of α and β into non-empty words. We thus get the desired result. \square

End of the proof of Proposition 5.3. We now suppose that U^\bullet is an alternal mould. If M^\bullet is an alternal mould, then

$$\tau(M^\bullet \circ U^\bullet) = (M^\bullet \otimes 1^\bullet + 1^\bullet \otimes M^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes 1^\bullet + 1^\bullet \otimes (M^\bullet \circ U^\bullet)$$

by (5.12)–(5.13), while, for M^\bullet symmetral,

$$\tau(M^\bullet \circ U^\bullet) = (M^\bullet \otimes M^\bullet) \circ U^\bullet = (M^\bullet \circ U^\bullet) \otimes (M^\bullet \circ U^\bullet).$$

Finally, if moreover U^\bullet is invertible for composition and $V^\bullet = (U^\bullet)^{\circ(-1)}$, then $\tau(V^\bullet) = \tau(V^\bullet) \circ (U^\bullet \circ V^\bullet) = (\tau(V^\bullet) \circ U^\bullet) \circ V^\bullet = \tau(V^\bullet \circ U^\bullet) \circ V^\bullet = (I^\bullet \otimes 1^\bullet + 1^\bullet \otimes I^\bullet) \circ V^\bullet = V^\bullet \otimes 1^\bullet + 1^\bullet \otimes V^\bullet$.

5.8 Proof of Proposition 5.4. Let M^\bullet be a symmetral mould. If D is induced by a derivation $d: \mathbf{A} \rightarrow \mathbf{A}$, then we can apply d to both sides of equation (5.2) and we get

$$\tau(DM^\bullet) = DM^\bullet \otimes M^\bullet + M^\bullet \otimes DM^\bullet. \quad (5.16)$$

Let us show that the same relation holds when $D = \nabla_{J^\bullet}$ with J^\bullet alternal (this includes the case $D = D_\varphi$). We set $C^\bullet = \nabla_{J^\bullet} M^\bullet$ and denote respectively by $\mathbf{J}^{\bullet,\bullet}, \mathbf{M}^{\bullet,\bullet}, \mathbf{C}^{\bullet,\bullet}$ the images of $J^\bullet, M^\bullet, C^\bullet$ by the homomorphism τ . We first observe that $C^\emptyset = 0$ and $\mathbf{C}^{\emptyset,\emptyset} = 0$. Let $\omega^1, \omega^2 \in \Omega^\bullet$ with at least one of them non-empty. From the definition of C^\bullet , we have

$$\begin{aligned} \mathbf{C}^{\omega^1, \omega^2} &= \sum_{\alpha, \beta, \gamma, \beta \neq \emptyset} \operatorname{sh}\left(\frac{\omega^1, \omega^2}{\alpha \cdot \beta \cdot \gamma}\right) M^{\alpha \cdot \|\beta\| \cdot \gamma} J^\beta \\ &= \sum_{\substack{\omega^1 = \alpha^1 \cdot \beta^1 \cdot \gamma^1 \\ \omega^2 = \alpha^2 \cdot \beta^2 \cdot \gamma^2}} \sum_{\substack{\alpha, \gamma \\ \beta \neq \emptyset}} \operatorname{sh}\left(\frac{\alpha^1, \alpha^2}{\alpha}\right) \operatorname{sh}\left(\frac{\gamma^1, \gamma^2}{\gamma}\right) M^{\alpha \cdot (\|\beta^1\| + \|\beta^2\|) \cdot \gamma} \operatorname{sh}\left(\frac{\beta^1, \beta^2}{\beta}\right) J^\beta \end{aligned}$$

by virtue of (5.14) with $s = 3$. The summation over β leads to the appearance of the factor $\mathbf{J}^{\beta^1, \beta^2}$. By altermality of J^\bullet , this factor vanishes if both β^1 and β^2 are non-empty, thus

$$\begin{aligned} \mathbf{C}^{\omega^1, \omega^2} &= \sum_{\substack{\omega^1 = \alpha^1 \cdot \beta^1 \cdot \gamma^1, \beta^1 \neq \emptyset}} \Phi_1(\alpha^1, \|\beta^1\|, \gamma^1; \omega^2) J^{\beta^1} \\ &\quad + \sum_{\substack{\omega^2 = \alpha^2 \cdot \beta^2 \cdot \gamma^2, \beta^2 \neq \emptyset}} \Phi_2(\omega^1; \alpha^2, \|\beta^2\|, \gamma^2) J^{\beta^2}, \end{aligned}$$

with $\Phi_1(\alpha^1, b, \gamma^1; \omega^2) = \sum_{\substack{\alpha, \gamma, \omega^2 = \alpha^2 \cdot \gamma^2}} \operatorname{sh}\left(\frac{\alpha^1, \alpha^2}{\alpha}\right) \operatorname{sh}\left(\frac{\gamma^1, \gamma^2}{\gamma}\right) M^{\alpha \cdot b \cdot \gamma}$ and a symmetric definition for Φ_2 . A moment of thought shows that

$$\Phi_1(\alpha^1, b, \gamma^1; \omega^2) = \sum_{\omega} \operatorname{sh}\left(\frac{\alpha^1 \cdot b \cdot \gamma^1, \omega^2}{\omega}\right) M^\omega = \mathbf{M}^{\alpha^1 \cdot b \cdot \gamma^1, \omega^2},$$

with a symmetric formula for Φ_2 , so that

$$\mathbf{C}^{\omega^1, \omega^2} = \sum_{\omega^1 = \alpha \cdot \beta \cdot \gamma} \mathbf{M}^{\alpha \cdot \|\beta\| \cdot \gamma, \omega^2} U^\beta + \sum_{\omega^2 = \alpha \cdot \beta \cdot \gamma} M^{\omega^1, \alpha \cdot \|\beta\| \cdot \gamma} U^\beta,$$

whence formula (5.16) follows.

Since the multiplicative inverse \widetilde{M}^\bullet of M^\bullet is known to be symmetral by Proposition 5.1, we can multiply both sides of (5.16) by $\tau(\widetilde{M}^\bullet)$ and use Lemma 5.1 and formula (5.7); this yields the symmetrality of $DM^\bullet \times \widetilde{M}^\bullet$ and $\widetilde{M}^\bullet \times DM^\bullet$.

5.9 There is a kind of converse to Proposition 5.4, which is essential in the application to the saddle-node; we state it in this context only:

Proposition 5.5. *Let $\Omega = \mathcal{N}$ as in (3.2) and $\mathbf{A} = \mathbb{C}[[x]]$. Then the mould \mathcal{V}^\bullet defined by Lemma 3.2 is symmetral.*

Proof. We must show

$$\mathcal{V}^\alpha \mathcal{V}^\beta = \sum_{\gamma \in \Omega^\bullet} \operatorname{sh}\left(\frac{\alpha, \beta}{\gamma}\right) \mathcal{V}^\gamma, \quad \alpha, \beta \in \Omega^\bullet. \quad (5.17)$$

Since $\mathcal{V}^\emptyset = 1$, this is obviously true for α or $\beta = \emptyset$. We now argue by induction on $r = r(\alpha) + r(\beta)$. We thus suppose $r \geq 1$ and, without loss of generality, both of α and β non-empty. With the notations $d = x^2 \frac{d}{dx}$, $\|\alpha\| = \alpha_1 + \cdots + \alpha_{r(\alpha)}$ and $\|\beta\| = \beta_1 + \cdots + \beta_{r(\beta)}$, we compute

$$\begin{aligned} A &:= (d + \|\alpha\| + \|\beta\|) \sum_{\gamma} \text{sh}(\alpha \cdot \gamma \cdot \beta) \mathcal{V}^\gamma \\ &= \sum_{\gamma \neq \emptyset} \text{sh}(\alpha \cdot \gamma \cdot \beta) (d + \|\gamma\|) \mathcal{V}^\gamma = \sum_{\gamma \neq \emptyset} \text{sh}(\alpha \cdot \gamma \cdot \beta) a_{\gamma_1} \mathcal{V}^{\cdot \gamma}, \end{aligned}$$

using the notation ' $\omega = (\omega_2, \dots, \omega_s)$ ' for any non-empty $\omega = (\omega_1, \dots, \omega_s)$ and the defining equation of \mathcal{V}^\bullet . Splitting the last summation according to the value of γ_1 , we get

$$A = \sum_{\delta} \text{sh}(\alpha \cdot \delta \cdot \beta) a_{\alpha_1} \mathcal{V}^\delta + \sum_{\delta} \text{sh}(\alpha \cdot \delta \cdot \beta) a_{\beta_1} \mathcal{V}^\delta = a_{\alpha_1} \mathcal{V}^{\cdot \alpha} \cdot \mathcal{V}^\beta + \mathcal{V}^\alpha \cdot a_{\beta_1} \mathcal{V}^{\cdot \beta}$$

(using the induction hypothesis), hence

$$A = (d + \|\alpha\|) \mathcal{V}^\alpha \cdot \mathcal{V}^\beta + \mathcal{V}^\alpha \cdot (d + \|\beta\|) \mathcal{V}^\beta = (d + \|\alpha\| + \|\beta\|) (\mathcal{V}^\alpha \mathcal{V}^\beta).$$

We conclude that both sides of (5.17) must coincide, because $d + \|\alpha\| + \|\beta\|$ is invertible if $\|\alpha\| + \|\beta\| \neq 0$ and both of them belong to $x\mathbb{C}[[x]]$, thus even if $\|\alpha\| + \|\beta\| = 0$ the desired conclusion holds. \square

6 General mould-comould expansions

6.1 We still assume that we are given a set Ω and a commutative \mathbb{C} -algebra \mathbf{A} . When Ω is the trivial one-element semigroup $\{0\}$, the algebra of \mathbf{A} -valued moulds on Ω is nothing but the algebra of formal series $\mathbf{A}[[T]]$, with its usual multiplication and composition laws: the monoid of words is then isomorphic to \mathbb{N} via the map r , and one can identify a mould M^\bullet with the generating series $\sum_{\omega \in \Omega^\bullet} M^\omega T^{r(\omega)}$; it is then easy to check that the above definitions of multiplication and composition boil down to the usual ones.

In the case of a general set Ω , the analogue of this is to identify a mould M^\bullet with the element $\sum M^{\omega_1, \dots, \omega_r} T_{\omega_1} \cdots T_{\omega_r}$ of the completion of the free associative (non-commutative) algebra generated by the symbols T_η , $\eta \in \Omega$. When replacing the T_η 's by elements B_η of an \mathbf{A} -algebra, one gets what is called a mould-comould expansion; we now define these objects in a context inspired by Section 3.

6.2 Suppose that $(\mathcal{F}, \text{val})$ is a complete pseudovaluation ring, possibly non-commutative, with unit denoted by Id , such that \mathcal{F} is also an \mathbf{A} -algebra. We thus have a ring homomorphism $\mu \in \mathbf{A} \mapsto \mu \text{Id} \in \mathcal{F}$, the image of which lies in the center of \mathcal{F} .

Definition 6.1. A comould on Ω with values in \mathcal{F} is any map $\mathbf{B}_\bullet: \omega \in \Omega^\bullet \mapsto \mathbf{B}_\omega \in \mathcal{F}$ such that $\mathbf{B}_\emptyset = \text{Id}$ and

$$\mathbf{B}_{\omega^1 \cdot \omega^2} = \mathbf{B}_{\omega^2} \mathbf{B}_{\omega^1}, \quad \omega^1, \omega^2 \in \Omega^\bullet. \quad (6.1)$$

Such an object could even be called *multiplicative comould* to emphasize that the map $\mathbf{B}_\bullet: \Omega^\bullet \rightarrow \mathcal{F}$ is required to be a monoid homomorphism from Ω^\bullet to the multiplicative monoid underlying the opposite ring of \mathcal{F} .

Observe that there is a one-to-one correspondence between comoulds and families $(B_\eta)_{\eta \in \Omega}$ of \mathcal{F} indexed by one-letter words: the formulas $\mathbf{B}_\emptyset = \text{Id}$ and $\mathbf{B}_\omega = B_{\omega_r} \cdots B_{\omega_1}$ for $\omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$ with $r \geq 1$ define a comould, which we call the *comould generated by $(B_\eta)_{\eta \in \Omega}$* , and all comoulds are obtained this way.

Suppose a comould \mathbf{B}_\bullet is given. For any \mathbf{A} -valued mould M^\bullet on Ω such that the family $(M^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable in \mathcal{F} (in particular this family has countable support—cf. Definition 3.2), we can consider the mould-comould expansion, also called *contraction of M^\bullet into \mathbf{B}_\bullet* ,

$$\sum M^\bullet \mathbf{B}_\bullet = \sum_{\omega \in \Omega^\bullet} M^\omega \mathbf{B}_\omega \in \mathcal{F}.$$

6.3 The example to keep in mind is related to Definition 3.3. Suppose that (\mathcal{A}, ν) is any complete pseudovaluation ring such that \mathcal{A} is a commutative \mathbf{A} -algebra, the unit of which is denoted by 1; thus \mathbf{A} is identified to a subalgebra of \mathcal{A} (for instance $(\mathcal{A}, \nu) = (\mathbb{C}[[x, y]], \nu_4)$ and $\mathbf{A} = \mathbb{C}[[x]]$). Denote by \mathcal{E} the subalgebra of $\text{End}_{\mathbb{C}}(\mathcal{A})$ consisting of operators having a valuation with respect to ν , so that $(\mathcal{E}, \text{val}_\nu)$ is a complete pseudovaluation ring. Let

$$\mathcal{F}_{\mathcal{A}, \mathbf{A}} = \{ \Theta \in \mathcal{E} \mid \Theta \text{ and } \mu \text{ Id commute for all } \mu \in \mathbf{A} \} = \mathcal{E} \cap \text{End}_{\mathbf{A}}(\mathcal{A}). \quad (6.2)$$

We get an \mathbf{A} -algebra, which is a closed subset of \mathcal{E} for the topology induced by val_ν , thus $(\mathcal{F}_{\mathcal{A}, \mathbf{A}}, \text{val}_\nu)$ is also a complete pseudovaluation ring; these are the \mathbf{A} -linear operators of \mathcal{A} having a valuation with respect to ν .

In practice, the B_η 's which generate a comould are related to the homogeneous components of an operator of \mathcal{A} that one wishes to analyse. In Section 3 for instance, the derivation $X - X_0$ of $\mathcal{A} = \mathbb{C}[[x, y]]$ was decomposed into a sum of multiples of B_n according to (3.3), where each term $a_n(x)B_n$ is homogeneous of degree n in the sense that it sends $y^{n_0}\mathbb{C}[[x]]$ in $y^{n_0+n}\mathbb{C}[[x]]$ for every n_0 . Observe that the commutation of the B_η 's with the image of $\mathbf{A} = \mathbb{C}[[x]]$ in \mathcal{E} reflects the fact that the vector field $X - X_0$ is “fibred” over the variable x ; similarly, one can look for a solution Θ of equation (3.1) in $\mathcal{F}_{\mathcal{A}, \mathbf{A}}$ because the corresponding formal transformation $(x, y) \mapsto \theta(x, y)$ is expected to be fibred likewise—cf. (2.5).

6.4 Returning to the general situation, we now show how, via mould-comould expansions, mould multiplication corresponds to multiplication in \mathcal{F} :

Proposition 6.1. Suppose that \mathbf{B}_\bullet is an \mathcal{F} -valued comould on Ω and that M^\bullet and N^\bullet are \mathbf{A} -valued moulds on Ω such that the families $(M^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and $(N^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ are formally summable. Then the mould $P^\bullet = M^\bullet \times N^\bullet$ gives

rise to a formally summable family $(P^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and

$$\sum (M^\bullet \times N^\bullet) \mathbf{B}_\bullet = \left(\sum N^\bullet \mathbf{B}_\bullet \right) \left(\sum M^\bullet \mathbf{B}_\bullet \right).$$

Proof. Let $\delta_* \in \mathbb{Z}$ such that $v_1(\omega) = \text{val}(M^\omega \mathbf{B}_\omega) \geq \delta_*$ and $v_2(\omega) = \text{val}(N^\omega \mathbf{B}_\omega) \geq \delta_*$ for all $\omega \in \Omega^\bullet$. Then

$$P^\omega \mathbf{B}_\omega = \sum_{\omega=\omega^1 \cdot \omega^2} N^{\omega^2} \mathbf{B}_{\omega^2} M^{\omega^1} \mathbf{B}_{\omega^1}$$

(since \mathbf{A} is a commutative algebra and its image in \mathcal{F} commutes with the \mathbf{B}_{ω^2} 's), thus $\text{val}(P^\omega \mathbf{B}_\omega) \geq \min\{v_1(\omega^1) + v_2(\omega^2) \mid \omega = \omega^1 \cdot \omega^2\} \geq 2\delta_*$ and, for any $\delta \in \mathbb{Z}$, the condition $\text{val}(P^\omega \mathbf{B}_\omega) \leq \delta$ implies that ω can be written as $\omega^1 \cdot \omega^2$ with $v_1(\omega^1) \leq \delta - \delta_*$ and $v_2(\omega^2) \leq \delta - \delta_*$, hence they are only finitely many such ω 's.

To compute $\sum P^\bullet \mathbf{B}_\bullet$, we can suppose Ω countable (replacing it, if necessary, by the set of all letters appearing in the union of the supports of $(M^\omega \mathbf{B}_\omega)$ and $(N^\omega \mathbf{B}_\omega)$, which is countable), choose an exhaustion of Ω by finite sets Ω_K , $K \geq 0$, and use $\Omega^{K,R} = \{\omega \in \Omega^\bullet \mid r = r(\omega) \leq R, \omega_1, \dots, \omega_r \in \Omega_K\}$, $K, R \geq 0$, as an exhaustion of Ω^\bullet . The conclusion follows from the identity

$$\begin{aligned} \left(\sum_{\omega \in \Omega^{K,R}} N^\omega \mathbf{B}_\omega \right) \left(\sum_{\omega \in \Omega^{K,R}} M^\omega \mathbf{B}_\omega \right) - \sum_{\omega \in \Omega^{K,R}} P^\omega \mathbf{B}_\omega \\ = \sum_{\substack{\omega^1, \omega^2 \in \Omega^{K,R} \\ r(\omega^1) + r(\omega^2) > R}} N^{\omega^2} \mathbf{B}_{\omega^2} M^{\omega^1} \mathbf{B}_{\omega^1}, \end{aligned}$$

where the right-hand side tends to 0 as $K, R \rightarrow \infty$, since its valuation is at least $\min\{v_1(\omega^1) + v_2(\omega^2) \mid \omega^1, \omega^2 \in \Omega^{K,R}, r(\omega^1) + r(\omega^2) > R\} \geq \nu_*(K, R) + \delta_*$, with

$$\nu_*(K, R) = \min\{\min(v_1(\omega), v_2(\omega)) \mid \omega \in \Omega^{K,R}, r(\omega) > R/2\} \xrightarrow[R \rightarrow \infty]{} \infty$$

for any K (because, for any finite subset F of Ω^\bullet , $\omega \notin F$ as soon as $r(\omega)$ is large enough). \square

6.5 Suppose Ω is a commutative semigroup. A motivation for the definition of mould composition in Section 4 is

Proposition 6.2. *Suppose that U^\bullet and M^\bullet are moulds such that the families $(U^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and*

$$\Theta_{\omega^1, \dots, \omega^s} = M^{\|\omega^1\|, \dots, \|\omega^s\|} U^{\omega^1} \cdots U^{\omega^s} \mathbf{B}_{\omega^1 \cdots \omega^s}, \quad s \geq 1, \omega^1, \dots, \omega^s \in \Omega^\bullet$$

are formally summable.¹⁰ Suppose moreover $U^\emptyset = 0$, let

$$B'_\eta = \sum_{\omega \in \Omega^\bullet \text{ s.t. } \|\omega\| = \eta} U^\omega \mathbf{B}_\omega, \quad \eta \in \Omega, \tag{6.3}$$

¹⁰Notice that the formal summability of the second family follows from the formal sumability of the first one when the valuation val on \mathcal{F} only takes non-negative values.

and we consider the comould \mathbf{B}'_\bullet generated by $\{B'_\eta, \eta \in \Omega\}$. Then the mould $C^\bullet = M^\bullet \circ U^\bullet$ gives rise to a formally summable family $(C^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and

$$\sum (M^\bullet \circ U^\bullet) \mathbf{B}_\bullet = \sum M^\bullet \mathbf{B}'_\bullet.$$

Proof. We have $C^\emptyset \mathbf{B}_\emptyset = M^\emptyset \mathbf{B}'_\emptyset$, since $C^\emptyset = M^\emptyset$. If ω and η are non-empty words in Ω^\bullet , with $\eta = (\eta_1, \dots, \eta_\sigma)$,

$$C^\omega \mathbf{B}_\omega = \sum_{\substack{s \geq 1, \omega^1, \dots, \omega^s \neq \emptyset \\ \omega = \omega^1 \cdots \omega^s}} \Theta_{\omega^1, \dots, \omega^s}, \quad M^\eta \mathbf{B}'_\eta = \sum_{\substack{\omega^1, \dots, \omega^\sigma \neq \emptyset \\ \|\omega^1\| = \eta_1, \dots, \|\omega^\sigma\| = \eta_\sigma}} \Theta_{\omega^1, \dots, \omega^\sigma}.$$

The conclusion follows easily. \square

The idea is that, when indexation by $\eta \in \Omega$ corresponds to a decomposition of an element of \mathcal{F} into homogeneous components, we use the mould U^\bullet to go from $X = \sum_{\eta \in \Omega} B_\eta$ to $Y = \sum_{\eta \in \Omega} B'_\eta$ by contracting it into the comould \mathbf{B}_\bullet associated with X ; then we use M^\bullet to go from Y to the contraction Z of M^\bullet into the comould \mathbf{B}'_\bullet associated with Y . Mould composition thus reflects the composition of these operations on elements of \mathcal{F} , $X \mapsto Y$ and $Y \mapsto Z$.

6.6 For example, suppose that $(\mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable. Then, in particular, $(B_\eta)_{\eta \in \Omega}$ is formally summable, and $X = \sum B_\eta$ is “exponentiable”: for any $t \in \mathbb{C}$, the series $\exp(tX) = \sum_{s \geq 0} \frac{t^s}{s!} X^s$ is convergent; moreover, $\exp(tX) = \sum \exp_t^\bullet \mathbf{B}_\bullet$. On the other hand, $\text{Id} + X$ has an “infinitesimal generator”: the series $Y = \sum_{s \geq 1} \frac{(-1)^{s-1}}{s} X^s$ is convergent and $\exp(Y) = \text{Id} + X$; one has $Y = \sum \log^\bullet \mathbf{B}_\bullet$.

Now, if $U^\bullet \in \mathfrak{L}^\bullet(\Omega, \mathbf{A})$ is such that $(U^{\omega^1} \cdots U^{\omega^s} \mathbf{B}_{\omega^1 \cdots \omega^s})_{s \geq 1, \omega^1, \dots, \omega^s \in \Omega^\bullet}$ is formally summable, then in particular $(U^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable and $X' = \sum U^\bullet \mathbf{B}_\bullet$ is exponentiable, with $\exp(tX') = \sum (\exp_t^\bullet \circ U^\bullet) \mathbf{B}_\bullet$ for any $t \in \mathbb{C}$.

Similarly, if $M^\bullet = 1^\bullet + V^\bullet \in G^\bullet(\Omega, \mathbf{A})$ with $(V^{\omega^1} \cdots V^{\omega^s} \mathbf{B}_{\omega^1 \cdots \omega^s})$ formally summable, then $\sum M^\bullet \mathbf{B}_\bullet$ has infinitesimal generator $\sum (\log^\bullet \circ V^\bullet) \mathbf{B}_\bullet$.

6.7 For the interpretation of the mould derivations ∇_{U^\bullet} defined by (4.2), consider a situation similar to that of Proposition 6.2, with a comould $\mathbf{B}_\bullet: \Omega^\bullet \rightarrow \mathcal{F}$, a mould $U^\bullet \in \mathfrak{L}^\bullet(\Omega, \mathbf{A})$ such that $(U^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable and, for each $\eta \in \Omega$, $B'_\eta \in \mathcal{F}$ still defined by (6.3). But instead of considering the comould \mathbf{B}'_\bullet generated by (B'_η) , i.e. $\mathbf{B}'_\omega = B'_{\omega_r} \cdots B'_{\omega_1}$ for $\omega = (\omega_1, \dots, \omega_r)$, set

$$\mathbf{B}'_\omega = \sum_{\omega = \alpha \beta \gamma, r(\beta)=1} \mathbf{B}_\gamma B'_\beta \mathbf{B}_\alpha, \quad \omega \in \Omega^\bullet$$

i.e. $\mathbf{B}'_\emptyset = 0$ and $\mathbf{B}'_\omega = \sum_{i=1}^r B_{\omega_r} \cdots B_{\omega_{i+1}} B'_{\omega_i} B_{\omega_{i-1}} \cdots B_{\omega_1}$ for $r \geq 1$ (beware that $\mathbf{B}'_\bullet: \Omega^\bullet \rightarrow \mathcal{F}$ is not a comould, since multiplicativity fails).

Then one can check the formal summability of $((\nabla_{U^\bullet} M^\omega) \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ for any mould M^\bullet such that the families $(M^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and $(M^{\alpha, \|\beta\|, \gamma} U^\beta \mathbf{B}_{\alpha, \beta, \gamma})_{\alpha, \beta, \gamma \in \Omega^\bullet}$ are formally summable, with

$$\sum (\nabla_{U^\bullet} M^\bullet) \mathbf{B}_\bullet = \sum M^\bullet \mathbf{B}'_\bullet.$$

If, moreover, there is an \mathbf{A} -linear derivation $\mathcal{D}: \mathcal{F} \rightarrow \mathcal{F}$ such that $\mathcal{D}B_\eta = B'_\eta$ for each $\eta \in \Omega$, then \mathbf{B}'_ω is nothing but $\mathcal{D}\mathbf{B}_\omega$ and the previous identity takes the form

$$\sum (\nabla_{U^\bullet} M^\bullet) \mathbf{B}_\bullet = \mathcal{D} \left(\sum M^\bullet \mathbf{B}_\bullet \right).$$

6.8 For a given commutative algebra \mathbf{A} , we now consider the case where

$$\Omega \subset \mathbb{Z}^n, \quad \mathcal{A} = \mathbf{A}[[y_1, \dots, y_n]],$$

for a fixed $n \in \mathbb{N}^*$.

Definition 6.2. Given $\eta \in \mathbb{Z}^n$ and $\Theta \in \text{End}_{\mathbf{A}}(\mathbf{A}[[y_1, \dots, y_n]])$, we say that Θ is homogeneous of degree η if $\Theta y^m \in \mathbf{A} y^{m+\eta}$ for every $m \in \mathbb{N}^n$ (with the usual notation $y^m = y_1^{m_1} \cdots y_n^{m_n}$ for monomials).

For example, any $\lambda \in \mathbf{A}^n$ gives rise to an operator

$$\mathcal{X}_\lambda = \lambda_1 y_1 \frac{\partial}{\partial y_1} + \cdots + \lambda_n y_n \frac{\partial}{\partial y_n} \quad (6.4)$$

which is homogeneous of degree 0, since $\mathcal{X}_\lambda y^m = \langle m, \lambda \rangle y^m$.

Suppose moreover that we are given a pseudovaluation $\text{val}: \mathcal{A} \rightarrow \mathbb{Z} \cup \{\infty\}$ such that $(\mathcal{A}, \text{val})$ is complete and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ are continuous, and consider $\mathcal{F} = \mathcal{F}_{\mathcal{A}, \mathbf{A}}$ as defined by (6.2). We suppose $\Omega \subset \mathbb{Z}^n$ because we are interested in \mathcal{F} -valued *homogeneous* comoulds, i.e. \mathcal{F} -valued comoulds \mathbf{B}_\bullet such that \mathbf{B}_ω is homogeneous of degree $\|\omega\| = \omega_1 + \cdots + \omega_r \in \mathbb{Z}^n$ for every non-empty $\omega \in \Omega^\bullet$, and homogeneous of degree 0 for $\omega = \emptyset$; in fact, the multiplicativity property (6.1) will not be used for what follows, the following proposition holds for any map $\mathbf{B}_\bullet: \Omega^\bullet \rightarrow \mathcal{F}$ provided it is homogeneous as just defined.

In the case of a comould satisfying the multiplicativity property as required in Definition 6.1, homogeneity is equivalent to the fact that each $B_\eta = \mathbf{B}_{(\eta)}$, $\eta \in \Omega$, is homogeneous of degree η .

Proposition 6.3. Let $\lambda \in \mathbf{A}^n$ and \mathbf{B}_\bullet be an \mathcal{F} -valued homogeneous comould. Then, for every \mathbf{A} -valued mould M^\bullet such that $(M^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable,

$$[\mathcal{X}_\lambda, \sum M^\bullet \mathbf{B}_\bullet] = \sum (D_\varphi M^\bullet) \mathbf{B}_\bullet, \quad \text{with } \varphi = \langle \cdot, \lambda \rangle: \Omega \rightarrow \mathbf{A}. \quad (6.5)$$

Thus, this mould derivation D_φ reflects the action of the derivation $\text{ad}_{\mathcal{X}_\lambda}$ of $\text{End}_{\mathbf{A}}(\mathcal{A})$.

Proof. We first check that, if $\Theta \in \text{End}_{\mathbf{A}}(\mathbf{A}[[y_1, \dots, y_n]])$ is homogeneous of degree $\eta \in \mathbb{Z}^n$, then

$$[\mathcal{X}_\lambda, \Theta] = \langle \eta, \lambda \rangle \Theta.$$

By \mathbf{A} -linearity and continuity, it is sufficient to check that both operators act the same way on a monomial y^m . We have $\Theta y^m = \beta_m y^{m+\eta}$ with a $\beta_m \in \mathbf{A}$, thus $\mathcal{X}_\lambda \Theta y^m = \langle m + \eta, \lambda \rangle \beta_m y^{m+\eta} = \langle m + \eta, \lambda \rangle \Theta y^m$ while $\Theta \mathcal{X}_\lambda y^m = \langle m, \lambda \rangle \Theta y^m$, hence $[\mathcal{X}_\lambda, \Theta] y^m = \langle \eta, \lambda \rangle \Theta y^m$ as required.

It follows that

$$[\mathcal{X}_\lambda, \mathbf{B}_\omega] = (\varphi(\omega_1) + \cdots + \varphi(\omega_r)) \mathbf{B}_\omega, \quad \omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet.$$

Let $N^\bullet = D_\varphi M^\bullet$. For any exhaustion of Ω^\bullet by finite sets I_k , letting $\Theta_k = \sum_{\omega \in I_k} M^\omega \mathbf{B}_\omega$ and $\Theta'_k = \sum_{\omega \in I_k} N^\omega \mathbf{B}_\omega$, we get $[\mathcal{X}_\lambda, \Theta_k] = \Theta'_k$. For every $f \in \mathcal{A}$, we have $\Theta'_k f \xrightarrow[k \rightarrow \infty]{} \sum N^\bullet \mathbf{B}_\bullet f$ on the one hand, while, by continuity of \mathcal{X}_λ , $[\mathcal{X}_\lambda, \Theta_k] f \xrightarrow[k \rightarrow \infty]{} [\mathcal{X}_\lambda, \sum M^\bullet \mathbf{B}_\bullet] f$ on the other hand. \square

6.9 Notice that, in the above situation, any \mathbb{C} -linear derivation $d: \mathbf{A} \rightarrow \mathbf{A}$ induces a derivation $\tilde{d}: \mathcal{A} \rightarrow \mathcal{A}$ (defined by $\tilde{d} \sum a_m y^m = \sum (da_m) y^m$) and a mould derivation D (defined just before Remark 4.1). If d commutes with the B_η , $\eta \in \Omega$, one easily gets

$$[\tilde{d}, \sum M^\bullet \mathbf{B}_\bullet] = \sum (DM^\bullet) \mathbf{B}_\bullet. \quad (6.6)$$

On the other hand, $D_\varphi M^\omega = \langle \|\omega\|, \lambda \rangle M^\omega$ if $\varphi = \langle \cdot, \lambda \rangle$. Thus $D_\varphi = \nabla$ when $n = 1$ and $\lambda = 1$. This is the relevant situation for the saddle-node:

Corollary 6.1. Choose $\Omega = \mathcal{N}$ as in (3.2), $\mathbf{A} = \mathbb{C}[[x]]$ and $(\mathcal{A}, \text{val}) = (\mathbf{A}[[y]], \nu_4)$, $\mathcal{F} = \mathcal{F}_{\mathcal{A}, \mathbf{A}}$. Let \mathbf{B}_\bullet denote the \mathcal{F} -valued comould generated by $B_\eta = y^{\eta+1} \frac{\partial}{\partial y}$. Let $(a_\eta)_{\eta \in \Omega}$ be as in (3.4) and

$$X_0 = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X = X_0 + \sum_{\eta \in \Omega} a_\eta B_\eta.$$

Then the mould-comould contraction $\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet \in \mathcal{F}$, where \mathcal{V}^\bullet is determined by Lemma 3.2, is solution of the conjugacy equation (3.1) $\Theta X = X_0 \Theta$ in $\text{End}_{\mathbb{C}}(\mathcal{A})$.

Proof. It was already observed that each B_η is homogeneous of degree η and the formal summability of $(\mathcal{V}^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ was checked in Lemma 3.3. Let $d = x^2 \frac{d}{dx}: \mathbf{A} \rightarrow \mathbf{A}$. The corresponding derivation of \mathcal{A} is $\tilde{d} = x^2 \frac{\partial}{\partial x}$, which commutes with the B_η 's. On the other hand, with the notation of Proposition 6.3, $X_0 = \tilde{d} + \mathcal{X}_1$. Since $X - X_0 = \sum J_a^\bullet \mathbf{B}_\bullet$ with the notation of Remark 4.1, equation (3.1) is equivalent to $[\tilde{d} + \mathcal{X}_1, \Theta] = \Theta \sum J_a^\bullet \mathbf{B}_\bullet$; plugging any formally convergent mould-comould expansion $\Theta = \sum M^\bullet \mathbf{B}_\bullet$ into it, we find $\sum (DM^\bullet + \nabla M^\bullet) \mathbf{B}_\bullet$ for the left-hand side by (6.5) and (6.6) while, according to Proposition 6.1, the right-hand side can be written $\sum (J_a^\bullet \times M^\bullet) \mathbf{B}_\bullet$, hence the conclusion follows from (4.5). \square

Remark 6.1. The symmetrality of the mould \mathcal{V}^\bullet obtained in Proposition 5.5 shows us that Θ is invertible, with inverse $\Theta^{-1} = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$. The proof of Theorem 1 will thus be complete when we have checked that Θ is an algebra automorphism; this will follow from the results of next section on the contraction of symmetrical moulds into a comould generated by derivations.

7 Contraction into a cosymmetral comould

7.1 For the interpretation of alternality and symmetrality of moulds in terms of the corresponding mould-comould expansions, we focus on the case where the comould \mathbf{B}_\bullet is generated by a family of \mathbf{A} -linear derivations $(B_\eta)_{\eta \in \Omega}$ of a commutative algebra \mathcal{A} .

The main result of this section is Proposition 7.1 below, according to which, in this case, *the contraction of an alternal mould into \mathbf{B}_\bullet gives rise to a derivation* and *the contraction of a symmetral mould gives rise to an algebra automorphism*.

7.2 We thus assume that (\mathcal{A}, ν) is a complete pseudovaluation ring such that \mathcal{A} is a commutative \mathbf{A} -algebra, and we define $\mathcal{F} = \mathcal{F}_{\mathcal{A}, \mathbf{A}}$ by (6.2). Since we shall be interested in the way the elements of \mathcal{F} act on products of elements of \mathcal{A} , we consider the left \mathcal{F} -module $\text{Bil}_{\mathbf{A}}$ of \mathbf{A} -bilinear maps from $\mathcal{A} \times \mathcal{A}$ to \mathcal{A} (with ring multiplication $(\Theta, \Phi) \in \mathcal{F} \times \text{Bil}_{\mathbf{A}} \mapsto \Theta \circ \Phi \in \text{Bil}_{\mathbf{A}}$) and its filtration

$$\mathcal{B}_\delta = \{ \Phi \in \text{Bil}_{\mathbf{A}} \mid \nu(\Phi(f, g)) \geq \nu(f) + \nu(g) + \delta \text{ for all } f, g \in \mathcal{A} \}, \quad \delta \in \mathbb{Z}.$$

By defining $\mathcal{B} = \bigcup_{\delta \in \mathbb{Z}} \mathcal{B}_\delta$ we get a left \mathcal{F} -submodule of $\text{Bil}_{\mathbf{A}}$, for which the filtration $(\mathcal{B}_\delta)_{\delta \in \mathbb{Z}}$ is exhaustive, separated and compatible with the filtration of \mathcal{F} induced by val_ν : the subgroups $\mathcal{F}_\delta = \{ \Theta \in \mathcal{F} \mid \text{val}_\nu(\Theta) \geq \delta \}$ satisfy $\mathcal{F}_\delta \mathcal{B}_{\delta'} \subset \mathcal{B}_{\delta+\delta'}$ for all $\delta, \delta' \in \mathbb{Z}$. The corresponding distance on \mathcal{B} is complete, by completeness of (\mathcal{A}, ν) .

We now define a map $\sigma: \mathcal{F} \rightarrow \mathcal{B}$ by $\Theta \in \mathcal{F} \mapsto \sigma(\Theta) = \Phi$ such that

$$\Phi: (f, g) \in \mathcal{A} \times \mathcal{A} \mapsto \Phi(f, g) = \Theta(fg) \in \mathcal{A}.$$

This map is to be understood as a kind of coproduct. Observe that σ is \mathcal{F} -linear, i.e. $\sigma(\Theta\Theta') = \Theta\sigma(\Theta')$ (thus it boils down to $\sigma(\Theta) = \Theta \circ \sigma(\text{Id})$, and $\sigma(\text{Id})$ is just the multiplication of \mathcal{A}) and continuous because $\sigma(\mathcal{F}_\delta) \subset \mathcal{B}_\delta$ for each $\delta \in \mathbb{Z}$.

Viewing \mathcal{F} as an \mathbf{A} -module, we also define an \mathbf{A} -linear map

$$\rho: \mathcal{F}_2 = \mathcal{F} \otimes_{\mathbf{A}} \mathcal{F} \rightarrow \mathcal{B}$$

by its action on decomposable elements:

$$\rho(\Theta_1 \otimes \Theta_2)(f, g) = (\Theta_1 f)(\Theta_2 g), \quad f, g \in \mathcal{A}$$

for any $\Theta_1, \Theta_2 \in \mathcal{F}$. (A remark parallel to the remark on Definition 5.3 applies: the kernel of ρ is the torsion submodule of \mathcal{F}_2 when \mathcal{A} is an integral domain; if moreover \mathbf{A} is principal, then ρ is injective.) Notice that, for $\Theta_1 \in \mathcal{F}_{\delta_1}$ and $\Theta_2 \in \mathcal{F}_{\delta_2}$, one has $\rho(\Theta_1 \otimes \Theta_2) \in \mathcal{B}_{\delta_1+\delta_2}$, hence the map $\tilde{\rho}: (\Theta_1, \Theta_2) \mapsto \rho(\Theta_1 \otimes \Theta_2)$ from $\mathcal{F} \times \mathcal{F}$ to \mathcal{B} is continuous.

Using the \mathbf{A} -algebra structure of \mathcal{F}_2 , we see that

$$\sigma(\Theta) = \rho(\xi), \quad \sigma(\Theta') = \rho(\xi') \quad \Rightarrow \quad \sigma(\Theta\Theta') = \rho(\xi\xi') \tag{7.1}$$

for any $\Theta, \Theta' \in \mathcal{F}$, $\xi, \xi' \in \mathcal{F}_2$.

7.3 With the above notations, the set of all \mathbf{A} -linear derivations of \mathcal{A} having a valuation is

$$\mathcal{L}_{\mathcal{F}} = \{ \Theta \in \mathcal{F} \mid \sigma(\Theta) = \rho(\Theta \otimes \text{Id} + \text{Id} \otimes \Theta) \}$$

(it is a Lie algebra for the bracketting $[\Theta_1, \Theta_2] = \Theta_1\Theta_2 - \Theta_2\Theta_1$). Letting \mathcal{F}^* denote the multiplicative group of invertible elements of \mathcal{F} , we may also consider its subgroup

$$G_{\mathcal{F}} = \{ \Theta \in \mathcal{F}^* \mid \sigma(\Theta) = \rho(\Theta \otimes \Theta) \},$$

the elements of which are \mathbf{A} -linear algebra automorphisms of \mathcal{A} .

Lemma 7.1. *Assume that the generators B_η , $\eta \in \Omega$, of an \mathcal{F} -valued comould \mathbf{B}_\bullet all belong to $\mathfrak{L}_{\mathcal{F}}$. Then*

$$\sigma(\mathbf{B}_\omega) = \sum_{\omega^1, \omega^2 \in \Omega^\bullet} \text{sh} \begin{pmatrix} \omega^1, \omega^2 \\ \omega \end{pmatrix} \rho(\mathbf{B}_{\omega^1} \otimes \mathbf{B}_{\omega^2}), \quad \omega \in \Omega^\bullet. \quad (7.2)$$

Such a comould is said to be *cosymmetral*.

Proof. Let $\omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$. We proceed by induction on r . Equation (7.2) holds if $r = 0$, since $\sigma(\text{Id}) = \rho(\text{Id} \otimes \text{Id})$, or $r = 1$ (by assumption); we thus suppose $r \geq 2$. By (6.1), we can write $\mathbf{B}_\omega = \mathbf{B}_{\cdot\omega} B_{\omega_1}$ with $\cdot\omega = (\omega_2, \dots, \omega_r)$. Using the induction hypothesis and (7.1), we get $\sigma(\mathbf{B}_\omega) = \rho(\xi)$ with

$$\begin{aligned} \xi &= \sum_{\alpha, \beta \in \Omega^\bullet} \text{sh} \begin{pmatrix} \alpha, \beta \\ \cdot\omega \end{pmatrix} (\mathbf{B}_\alpha \otimes \mathbf{B}_\beta)(B_{\omega_1} \otimes \text{Id} + \text{Id} \otimes B_{\omega_1}) \\ &= \sum_{\alpha, \beta \in \Omega^\bullet} \text{sh} \begin{pmatrix} \alpha, \beta \\ \cdot\omega \end{pmatrix} (\mathbf{B}_{\omega_1 \cdot \alpha} \otimes \mathbf{B}_\beta + \mathbf{B}_\alpha \otimes \mathbf{B}_{\omega_1 \cdot \beta}). \end{aligned}$$

This coincides with $\sum_{\alpha, \beta \in \Omega^\bullet} \text{sh} \begin{pmatrix} \alpha, \beta \\ \omega \end{pmatrix} \mathbf{B}_\alpha \otimes \mathbf{B}_\beta$, since

$$\text{sh} \begin{pmatrix} \alpha, \beta \\ \omega \end{pmatrix} = \text{sh} \begin{pmatrix} \alpha, \beta \\ \cdot\omega \end{pmatrix} 1_{\{\alpha_1 = \omega_1\}} + \text{sh} \begin{pmatrix} \alpha, \beta \\ \cdot\omega \end{pmatrix} 1_{\{\beta_1 = \omega_1\}}$$

(particular case of (5.6) with $\gamma^1 = \omega_1$ and $\gamma^2 = \cdot\omega$). \square

7.4 We are now ready to study the effect of alternality or symmetrality in this context.

Proposition 7.1. *Suppose that \mathbf{B}_\bullet is an \mathcal{F} -valued cosymmetral comould and let $M^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$ be such that $(M^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is formally summable. Then:*

- If M^\bullet is alternal, then $\sum M^\bullet \mathbf{B}_\bullet \in \mathfrak{L}_{\mathcal{F}}$.
- If M^\bullet is symmetral, then $\sum M^\bullet \mathbf{B}_\bullet \in G_{\mathcal{F}}$.
- More generally, denoting by $\mathbf{M}^{\bullet\bullet}$ the image of M^\bullet by the homomorphism τ of Lemma 5.1 and assuming that the family $(\rho(\mathbf{M}^{\alpha, \beta} \mathbf{B}_\alpha \otimes \mathbf{B}_\beta))_{(\alpha, \beta) \in \Omega^\bullet \times \Omega^\bullet}$ is formally summable in \mathcal{B} ,

$$\sigma \left(\sum M^\bullet \mathbf{B}_\bullet \right) = \sum_{(\alpha, \beta) \in \Omega^\bullet \times \Omega^\bullet} \rho(\mathbf{M}^{\alpha, \beta} \mathbf{B}_\alpha \otimes \mathbf{B}_\beta). \quad (7.3)$$

Proof. Let $\delta_* \in \mathbb{Z}$ such that $v(\omega) = \text{val}_\nu(M^\omega \mathbf{B}_\omega) \geq \delta_*$ for all $\omega \in \Omega^\bullet$ and $\Theta = \sum M^\bullet \mathbf{B}_\bullet$. We shall use the notation $\Phi_{\alpha,\beta} = M^{\alpha,\beta} \mathbf{B}_\alpha \otimes \mathbf{B}_\beta$ for $(\alpha, \beta) \in \Omega^\bullet \times \Omega^\bullet$. Lemma 5.2 yields

$$\Phi_{\alpha,\beta} = 1_{\{\beta=\emptyset\}} M^\alpha \mathbf{B}_\alpha \otimes \text{Id} + 1_{\{\alpha=\emptyset\}} \text{Id} \otimes M^\beta \mathbf{B}_\beta$$

for M^\bullet alternal and $\Phi_{\alpha,\beta} = M^\alpha \mathbf{B}_\alpha \otimes M^\beta \mathbf{B}_\beta$ for M^\bullet symmetral. In both cases, the set $\{(\alpha, \beta) \in \Omega^\bullet \times \Omega^\bullet \mid \rho(\Phi_{\alpha,\beta}) \notin \mathcal{B}_\delta\}$ is thus finite for any $\delta \in \mathbb{Z}$, in view of the formal summability hypothesis, and the sum of the family $(\rho(\Phi_{\alpha,\beta}))$ is respectively $\rho(\Theta \otimes \text{Id} + \text{Id} \otimes \Theta)$ or $\rho(\Theta \otimes \Theta)$, by continuity of $\tilde{\rho}$. Therefore it is sufficient to prove the third property (the invertibility of Θ when M^\bullet is symmetral is a simple consequence of Proposition 6.1 and of the invertibility of M^\bullet).

We thus assume $(\rho(\Phi_{\alpha,\beta}))$ formally summable in \mathcal{B} . As in the proof of Proposition 6.1, we can suppose Ω countable and choose an exhaustion of Ω^\bullet by finite sets of the form $\Omega^{K,R}$. Then, by virtue of the definition of τ and of Lemma 7.1,

$$\begin{aligned} A_{K,R} &:= \sum_{(\alpha,\beta) \in \Omega^{K,R} \times \Omega^{K,R}} \rho(\Phi_{\alpha,\beta}) - \sigma \left(\sum_{\omega \in \Omega^{K,R}} M^\omega \mathbf{B}_\omega \right) \\ &= \left(\sum_{\alpha,\beta \in \Omega^{K,R}, \omega \in \Omega^\bullet} - \sum_{\alpha,\beta \in \Omega^\bullet, \omega \in \Omega^{K,R}} \right) \text{sh} \left(\frac{\alpha, \beta}{\omega} \right) M^\omega \rho(\mathbf{B}_\alpha \otimes \mathbf{B}_\beta) \\ &= \sum_{\substack{\alpha, \beta \in \Omega^{K,R} \\ \text{s.t. } r(\alpha) + r(\beta) > R}} \rho(\Phi_{\alpha,\beta}) \end{aligned}$$

(the last equality stems from the fact that, if $\omega \in \Omega^{K,R}$, then $\text{sh} \left(\frac{\alpha, \beta}{\omega} \right) \neq 0$ implies $\alpha, \beta \in \Omega^{K,R}$). The formal summability of $(\rho(\Phi_{\alpha,\beta}))$ yields $A_{K,R} \rightarrow 0$ as $K, R \rightarrow \infty$, which is the desired result since σ is continuous. \square

Remark 7.1. The proof of Theorem 1 is now complete: in view of the symmetry of \mathcal{V}^\bullet with $\Omega = \mathbb{N}$ and $\mathbf{A} = \mathbb{C}[[x]]$ (Proposition 5.5) and the cosymmetry of \mathbf{B}_\bullet defined by (3.5) with $\mathcal{F} = \mathcal{F}_{\mathcal{A}, \mathbf{A}}$, $(\mathcal{A}, \text{val}) = (\mathbb{C}[[x, y]], \nu_4)$, Proposition 7.1 shows that $\Theta = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet$ is an automorphism of \mathcal{A} . As noticed in Remark 6.1, this was the only thing which remained to be checked.

7.5 Another way of checking that the contraction of an alternal mould into a cosymmetrical comould \mathbf{B}_\bullet is a derivation is to express it as a sum of iterated Lie brackets of the derivations B_η which generate the comould.

For $\omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$ with $r \geq 2$, let

$$\mathbf{B}_{[\omega]} = [B_{\omega_r}, [B_{\omega_{r-1}}, [\dots [B_{\omega_2}, B_{\omega_1}] \dots]]].$$

One can check that, for any alternal mould M^\bullet and for any finite subset Ω_f of Ω ,

$$\sum_{\omega \in \Omega_f^r} M^\omega \mathbf{B}_\omega = \frac{1}{r} \sum_{\omega \in \Omega_f^r} M^\omega \mathbf{B}_{[\omega]}, \quad r \geq 2$$

(identifying Ω_f^r with the sets of all words of length r the letters of which belong to Ω_f). The proof is left to the reader.

7.6 Let \mathbf{B}_\bullet denote an \mathcal{F} -valued comould. Suppose that $(B_\eta)_{\eta \in \Omega}$ is formally summable and consider $Y = \sum_{\eta \in \Omega} B_\eta \in \mathcal{F}$.

We have seen that, by definition, the comould is cosymmetral iff each B_η is a derivation of \mathcal{A} ; then Y is itself a derivation. This is the situation when there is an appropriate notion of homogeneity, as in Definition 6.2, and we expand an \mathbf{A} -linear derivation Y into a sum of homogeneous components, each B_η being homogeneous of degree η .

Suppose now that the object to analyse is not a singular vector field, as in the case of the saddle-node, but a local transformation; considering the associated substitution operator, we are thus led to an automorphism of \mathcal{A} , typically of the form $\phi = \text{Id} + Y$. Then the homogeneous components B_η of Y are no longer derivations; expanding $\sigma(\phi) = \rho(\phi \otimes \phi)$, we rather get

$$\sigma(B_\eta) = \rho \left(B_\eta \otimes \text{Id} + \sum_{\eta' + \eta'' = \eta} B_{\eta'} \otimes B_{\eta''} + \text{Id} \otimes B_\eta \right). \quad (7.4)$$

The comould \mathbf{B}_\bullet they generate is then called *cosymmetrel*. A cosymmetrel comould is characterized by identities similar to (7.2) but with the shuffling coefficients $\text{sh}(\omega_1 \omega_2)$ replaced by new ones, denoted by $\text{ctsh}(\omega_1 \omega_2)$ and called “contracting shuffling coefficients”.

Dually, using these new coefficients instead of the previous shuffling coefficients in formulas (5.1) and (5.2), one gets the definition of *alternel* and *symmetrel* moulds, which were only briefly alluded to at the beginning of Section 5.

The contraction of alternel or symmetrel moulds into cosymmetrel comoulds enjoy properties parallel to those that we just described in the cosymmetral case. This allows one to treat local vector fields and local discrete dynamical systems with completely parallel formalisms.

Part C: Resurgence, alien calculus and other applications

8 Resurgence of the normalising series

8.1 The purpose of this section is to use the mould-comould representation of the formal normalisation of the saddle-node given by Theorem 1 to deduce “resurgent properties”. We begin by a few reminders about Écalle’s Resurgence theory. We follow the notations of [14].

– The formal Borel transform is the \mathbb{C} -linear homomorphism

$$\mathcal{B} : \tilde{\varphi}(z) = \sum_{n \geq 0} c_n z^{-n-1} \in z^{-1} \mathbb{C}[[z^{-1}]] \mapsto \hat{\varphi}(\zeta) = \sum_{n \geq 0} c_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]].$$

In the case of a convergent $\tilde{\varphi}$, one gets a convergent series $\widehat{\varphi}$ which defines an entire function of exponential type. Namely, if $\varphi(x) = \tilde{\varphi}(-1/x) \in x\mathbb{C}[[x]]$ has radius of convergence $> \rho$, then there exists $K > 0$ such that $|\varphi(x)| \leq K|x|$ for $|x| \leq \rho$ and this implies, by virtue of the Cauchy inequalities, that $|c_n| \leq K\rho^{-n}$, hence $\widehat{\varphi}$ is entire and

$$|\widehat{\varphi}(\zeta)| \leq K e^{\rho^{-1}|\zeta|}, \quad \zeta \in \mathbb{C}. \quad (8.1)$$

We are particularly interested in the case where $\widehat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ without being necessarily entire; this is equivalent to Gevrey-1 growth for the coefficients of $\tilde{\varphi}$:

$$\begin{aligned} \tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1 &\iff \exists C, K > 0 \text{ s.t. } |c_n| \leq KC^n n! \text{ for all } n \\ &\iff \mathcal{B}\tilde{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}. \end{aligned}$$

– The counterpart in $\mathbb{C}[[\zeta]]$ of the multiplication (Cauchy product) of $z^{-1}\mathbb{C}[[z^{-1}]]$ is called *convolution* and denoted by $*$, thus $\mathcal{B}(\tilde{\varphi} \cdot \tilde{\psi}) = \mathcal{B}(\tilde{\varphi}) * \mathcal{B}(\tilde{\psi})$. Now, if $\widehat{\varphi} = \mathcal{B}(\tilde{\varphi})$ and $\widehat{\psi} = \mathcal{B}(\tilde{\psi})$ belong to $\mathbb{C}\{\zeta\}$, then $\widehat{\varphi} * \widehat{\psi} \in \mathbb{C}\{\zeta\}$ and this germ of holomorphic function is determined by

$$(\widehat{\varphi} * \widehat{\psi})(\zeta) = \int_0^\zeta \widehat{\varphi}(\zeta_1) \widehat{\psi}(\zeta - \zeta_1) \quad \text{for } |\zeta| \text{ small enough.} \quad (8.2)$$

We have an algebra $(\mathbb{C}\{\zeta\}, *)$ without unit, isomorphic via \mathcal{B} to $(z^{-1}\mathbb{C}[[z^{-1}]]_1, \cdot)$. By adjunction of unit, we get an algebra isomorphism

$$\mathcal{B}: \mathbb{C}[[z^{-1}]]_1 \xrightarrow{\sim} \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$$

(where $\delta = \mathcal{B}1$ is a symbol for the unit of convolution). We can even take into account the differential $\frac{d}{dz}$: its counterpart via \mathcal{B} is $\widehat{\partial}: c\delta + \widehat{\varphi}(\zeta) \mapsto -\zeta\widehat{\varphi}(\zeta)$.

– Let us now consider all the rectifiable oriented paths of \mathbb{C} which start from the origin and then avoid \mathbb{Z} , i.e. oriented paths represented by absolutely continuous maps $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \mathbb{Z}^*$ such that $\gamma(0) = 0$ and $\gamma^{-1}(0)$ is connected. We denote by $\mathcal{R}(\mathbb{Z})$ the set of all homotopy classes $[\gamma]$ of such paths γ and by π the map $[\gamma] \in \mathcal{R}(\mathbb{Z}) \mapsto \gamma(1) \in \mathbb{C} \setminus \mathbb{Z}^*$; considering π as a covering map, we get a Riemann surface structure on $\mathcal{R}(\mathbb{Z})$.

Observe that $\pi^{-1}(0)$ consists of a single point, the “origin” of $\mathcal{R}(\mathbb{Z})$; this is the only difference between $\mathcal{R}(\mathbb{Z})$ and the universal cover of $\mathbb{C} \setminus \mathbb{Z}$. The space $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ of all holomorphic functions of $\mathcal{R}(\mathbb{Z})$ can be identified with the space of all $\widehat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ which admit an analytic continuation along any representative of any element of $\mathcal{R}(\mathbb{Z})$ (cf. [14], Definition 3 and Lemma 2).

Definition 8.1. We define the convolutive model of the algebra of resurgent functions over \mathbb{Z} as $\widehat{\mathbf{R}}_{\mathbb{Z}} = \mathbb{C}\delta \oplus \widehat{H}(\mathcal{R}(\mathbb{Z}))$. We define the formal model of the algebra of resurgent functions over \mathbb{Z} as $\widetilde{\mathbf{R}}_{\mathbb{Z}} = \mathcal{B}^{-1}(\widehat{\mathbf{R}}_{\mathbb{Z}})$.

It turns out that $\widehat{\mathbf{R}}_{\mathbb{Z}}$ is a subalgebra of the convolution algebra $\mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$, i.e. the aforementioned property of analytic continuation is stable by convolution; the proof of this fact relies on the notion of symmetrically contractile path (see

for instance *op. cit.*, §1.3), which we shall not develop here. Therefore $\tilde{\mathbf{R}}_{\mathbb{Z}}$ is a subalgebra of $\mathbb{C}[[z^{-1}]]$ and \mathcal{B} induces an algebra isomorphism $\tilde{\mathbf{R}}_{\mathbb{Z}} \rightarrow \widehat{\mathbf{R}}_{\mathbb{Z}}$.

– An obvious example of element of $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ is an entire function, or a meromorphic function of \mathbb{C} without poles outside \mathbb{Z}^* . Indeed, for such a function $\widehat{\varphi}$, we can define $\widehat{\phi} \in \widehat{H}(\mathcal{R}(\mathbb{Z}))$ by $\widehat{\phi}(\zeta) = \widehat{\varphi}(\pi(\zeta))$ for all $\zeta \in \mathcal{R}(\mathbb{Z})$. We usually identify $\widehat{\varphi}$ and $\widehat{\phi}$.

For example, if $\omega_1 \neq 0$, Remark 3.1 shows that $\tilde{\mathcal{V}}^{\omega_1}(z) = \mathcal{V}^{\omega_1}(-1/z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ has formal Borel transform $\widehat{\mathcal{V}}^{\omega_1}(\zeta) = \sum \omega_1^{-r-1} (-\widehat{\partial})^r \widehat{a}_{\omega_1} = \frac{\widehat{a}_{\omega_1}(\zeta)}{\omega_1 - \zeta}$, where \widehat{a}_{ω_1} denotes the formal Borel transform of $a_{\omega_1}(-1/z)$, which is an entire function (since a_{ω_1} is convergent), thus $\widehat{\mathcal{V}}^{\omega_1}$ is meromorphic with at most one simple pole, located at ω_1 . On the other hand, $\widehat{\mathcal{V}}^0(\zeta) = \frac{1}{\zeta} \widehat{a}_0(\zeta)$ is entire.

We shall see that, for each non-empty word ω , the formal Borel transform of $\mathcal{V}^{\omega}(-1/z)$ belongs to $\widehat{H}(\mathcal{R}(\mathbb{Z}))$, but this function is usually not meromorphic if $r(\omega) \geq 2$. For instance, for $\omega = (\omega_1, \omega_2)$, one gets $\frac{1}{-\zeta + \omega_1 + \omega_2} (\widehat{a}_{\omega_1} * \widehat{\mathcal{V}}^{\omega_2})$ which is multivalued in general (see formula (8.9) below for the general case).

– A formal series $\tilde{\varphi}(z)$ without constant term belongs to $\tilde{\mathbf{R}}_{\mathbb{Z}}$ iff its formal Borel transform $\widehat{\varphi}(\zeta)$ converges to a germ of holomorphic function which extends analytically to $\mathcal{R}(\mathbb{Z})$. In particular, the principal branch¹¹ of $\widehat{\varphi}$ is holomorphic in sectors which extend up to infinity. If it has at most exponential growth in a sector $\{\zeta \in \mathbb{C} \mid \theta_1 \leq \arg \zeta \leq \theta_2\}$ (as is the case of $\widehat{\mathcal{V}}^{\omega_1}(\zeta)$ for instance), then one can perform a Laplace transform and get a function

$$\tilde{\varphi}^{\text{ana}}(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}(\zeta) e^{-z\zeta} d\zeta, \quad \theta \in [\theta_1, \theta_2],$$

which is analytic for z belonging to a sectorial neighbourhood of infinity. This is called *Borel-Laplace summation* (see e.g. [14], §1.1).

Since multiplication is turned into convolution by \mathcal{B} and then turned again into multiplication by the Laplace transform, and similarly with $\frac{d}{dz}$ which is transformed into multiplication by $-\zeta$ by \mathcal{B} , the Borel-Laplace process transforms the formal solution of a differential equation like (2.8), (2.9) or (3.7) into an analytic solution of the same equation.

8.2 The stability of $\tilde{\mathbf{R}}_{\mathbb{Z}}$ under multiplication together with the previous computation explains to some extent why we can expect the solutions of a non-linear problem like the formal classification of the saddle-node to be resurgent. However, controlling products in $\tilde{\mathbf{R}}_{\mathbb{Z}}$ means controlling convolution products in $\widehat{\mathbf{R}}_{\mathbb{Z}}$, and it is not so easy to extract from the stability statement the quantitative information which would guarantee the convergence in $\tilde{\mathbf{R}}_{\mathbb{Z}}$ of a method of majorant series for instance (see the discussion at the end of the sketch of proof of Theorem 2 of [14]).

Thanks to the mould-comould expansion given in Section 3, we shall be able to use much simpler arguments: the convolution product of an element of $\widehat{\mathbf{R}}_{\mathbb{Z}}$ with an

¹¹The principal branch is defined as the analytic continuation of $\widehat{\varphi}$ in the maximal open subset of \mathbb{C} which is star-shaped with respect to 0; its domain is the cut plane obtained by removing the singular half-lines $[1, +\infty[$ and $]-\infty, -1]$, unless $\widehat{\varphi}$ happens to be regular at 1 or -1 .

entire function belongs to $\widehat{\mathbf{R}}_{\mathbb{Z}}$ and efficient bounds are available in this particular case of the stability statement (much easier than the general one—see Lemma 8.3 below).

Theorem 2. *Consider the saddle-node problem, with hypotheses (2.1)–(2.2). Let $\theta(x, y) = \left(x, y + \sum_{n \geq 0} \varphi_n(x)y^n\right)$ denote the formal transformation, the substitution operator of which is $\Theta = \sum \mathcal{V}^{\bullet} \mathbf{B}_{\bullet}$, in accordance with Lemmas 3.2 and 3.3 and Theorem 1. Let $\theta^{-1}(x, y) = \left(x, y + \sum_{n \geq 0} \psi_n(x)y^n\right)$ denote the inverse transformation.*

Then, for each $n \in \mathbb{N}$, the formal series $\tilde{\varphi}_n(z) = \varphi_n(-1/z)$ and $\tilde{\psi}_n(z) = \psi_n(-1/z)$ belong to $\widetilde{\mathbf{R}}_{\mathbb{Z}}$, and the analytic continuation of the formal Borel transforms $\widehat{\varphi}_n(\zeta), \widehat{\psi}_n(\zeta) \in \mathbb{C}\{\zeta\}$ satisfy the following:

- (i) *All the branches of the analytic continuation of $\widehat{\varphi}_n$ are regular at the points of $n + \mathbb{N} = \{n, n+1, n+2, \dots\}$.*
- (ii) *All the branches of the analytic continuation of $\widehat{\psi}_n$ are regular at the points of $-\mathbb{N}^* = \{-1, -2, -3, \dots\}$, with the sole exception that the branches of $\widehat{\psi}_0$ may have simple poles at -1 .*
- (iii) *Given any $\rho \in]0, \frac{1}{2}[$ and $N \in \mathbb{N}^*$, there exist positive constants K, L, C which depend only on ρ, N such that, for any $(\rho, N, n - \mathbb{N}^*)$ -adapted infinite path γ issuing from the origin,*

$$|\widehat{\varphi}_n(\gamma(t))| \leq KL^n e^{(n^2+1)Ct} \text{ for all } t \geq 0 \text{ and } n \in \mathbb{N}, \quad (8.3)$$

while, for any (ρ, N, \mathbb{N}) -adapted infinite path γ issuing from the origin,

$$|\widehat{\psi}_n(\gamma(t))| \leq KL^n e^{Ct} \text{ for } n \geq 1, \quad |(\gamma(t) + 1)\widehat{\psi}_0(\gamma(t))| \leq K e^{Ct} \quad (8.4)$$

for all $t \geq 0$.

What we call $(\rho, N, \mathcal{P}^{\pm})$ -adapted infinite path, with $\mathcal{P}^+ = \mathbb{N}$ or $\mathcal{P}^- = n - \mathbb{N}^*$, is defined below in Definition 8.2; see Figure 1. These are arc-length parametrised paths $\gamma: [0, +\infty[\rightarrow \mathbb{C}$ (i.e. γ is absolutely continuous and $|\dot{\gamma}(t)| = 1$ for almost every t) which start as rectilinear segments of length ρ issuing from the origin and which then do not approach \mathcal{P}^{\pm} nor $\pm\Sigma(\rho, N)$ at a distance $< \rho$, where $\pm\Sigma(\rho, N)$ denotes the sector of half-opening $\arcsin(\rho/N)$ bisected by $\pm[N, +\infty[$.

In particular, inequalities (8.3)–(8.4) yield an exponential bound at infinity for the principal branch of each $\widehat{\varphi}_n$ or $\widehat{\psi}_n$ along all the half-lines issuing from 0 except the singular half-lines $\pm[0, +\infty[$ (the half-line $[0, +\infty[$ is not singular for $\widehat{\varphi}_0$ and the half-line $-[0, +\infty[$ is not singular for any $\widehat{\psi}_n$).

We recall that Θ establishes a conjugacy between the saddle-node vector field X and its normal form X_0 , thus the formal series $\tilde{\varphi}_n(z)$ are the components of a formal integral $\tilde{Y}(z, u)$, as described in (2.6)–(2.7). The resurgence statement contained in Theorem 2 thus means that the formal solutions of the singular differential equations (2.8)–(2.9) may be divergent but that this divergence is of a very precise nature. We shall briefly indicate in Section 10 how alien calculus

allows one to take advantage of this information to study the problem of analytic classification.

Remark 8.1. Theorem 2 also permits the obtention of analytic solutions of (2.8)–(2.9) via Borel-Laplace summation. It is thus worth mentioning that one can get rid of the dependence on n in the exponential which appears in (8.3), provided one restricts oneself to paths which start from the origin and then do not approach at a distance $< \rho$ the set $\mathbb{Z} \cup \Sigma(\rho, N) \cup (-\Sigma(\rho, N))$, and which cross the cuts (the segments between consecutive points of \mathbb{Z}) at most N' times. For instance, with $N' = 0$, one obtains

$$|\widehat{\varphi}_n(\zeta)| \leq KL^n e^{C|\zeta|} \quad (8.5)$$

for the principal branch of $\widehat{\varphi}_n$, possibly with larger constants K, L, C but still independent of n . For the other branches, which correspond to $N' \geq 1$ and $N \geq 2$, one has to resort to symmetrically contractile paths and the implied constants K, L, C depend only on ρ, N, N' .

Therefore, when performing Laplace transform, inequalities (8.5) allow one to get the same domain of analyticity for all the functions $\widehat{\varphi}_n^{\text{ana}}(z)$ solutions of (2.8)–(2.9) (a sectorial neighbourhood of infinity which depends only on C ; see e.g. [14], §1.1), with explicit bounds which make it possible to study the domain of analyticity of a *sectorial formal integral* $\widetilde{Y}^{\text{ana}}(z, u) = u e^z + \sum u^n e^{nz} \widehat{\varphi}_n^{\text{ana}}(z)$ or of analytic normalising transformations $\varphi^{\text{ana}}(x, y), \psi^{\text{ana}}(x, y)$.

This will be used in Section 11.

The rest of this section is devoted to the proof of Theorem 2 and to the derivation of inequalities (8.5).

8.3 Using $\Omega = \mathcal{N} = \{ \eta \in \mathbb{Z} \mid \eta \geq -1 \}$ as an alphabet, we know that the $\mathbb{C}[[x]]$ -valued moulds \mathcal{V}^\bullet and \mathcal{V}^\bullet are symmetral and mutually inverse for mould multiplication. We recall that

$$\Theta^{-1} = \sum \mathcal{V}^\bullet \mathbf{B}_\bullet, \quad \mathcal{V}^{\omega_1, \dots, \omega_r} = (-1)^r \mathcal{V}^{\omega_r, \dots, \omega_1}.$$

With the notation $\|\omega\| = \omega_1 + \dots + \omega_r$ for any non-empty word $\omega \in \Omega^\bullet$, equation (3.11) can be written $\mathbf{B}_\omega y = \beta_\omega y^{\|\omega\|+1}$, with the coefficients β_ω defined at the end of Section 3. As was already observed, since $\Theta y = \sum \varphi_n(x) y^n$ and $\Theta^{-1} y = \sum \psi_n(x) y^n$, the formal series we are interested in can be written as formally convergent series in $\mathbb{C}[[x]]$:

$$\varphi_n = \sum_{\|\omega\|=n-1} \beta_\omega \mathcal{V}^\omega, \quad \psi_n = \sum_{\|\omega\|=n-1} \beta_\omega \mathcal{V}^\omega, \quad n \in \mathbb{N}, \quad (8.6)$$

with summation over all words ω of positive length subject to the condition $\|\omega\| = n-1$. In fact, not all of these words contribute in these series:

Lemma 8.1. *For any non-empty $\omega = (\omega_1, \dots, \omega_r) \in \Omega^\bullet$, using the notations*

$$\check{\omega}_i = \omega_1 + \dots + \omega_i, \quad \hat{\omega}_i = \omega_i + \dots + \omega_r, \quad 1 \leq i \leq r, \quad (8.7)$$

we have

$$\beta_\omega \neq 0 \Rightarrow \|\omega\| \geq -1, \quad \check{\omega}_1, \dots, \check{\omega}_{r-1} \geq 0 \text{ and } \hat{\omega}_1, \dots, \hat{\omega}_r \leq \|\omega\|.$$

Proof. We have

$$\beta_\omega = 1 \text{ if } r = 1, \quad \beta_\omega = (\check{\omega}_1 + 1)(\check{\omega}_2 + 1) \cdots (\check{\omega}_{r-1} + 1) \text{ if } r \geq 2. \quad (8.8)$$

The property $\beta_\omega \neq 0 \Rightarrow \|\omega\| \geq -1$ was already observed at the end of Section 3, as a consequence of $\mathbf{B}_\omega y \in \mathbb{C}[[y]]$ (one can also argue directly from formula (8.8)).

Now suppose $\beta_\omega \neq 0$ and $1 \leq i \leq r-1$. The identity

$$\beta_\omega = \beta_{\omega_1, \dots, \omega_i} (\check{\omega}_i + 1) \cdots (\check{\omega}_{r-1} + 1)$$

implies $\check{\omega}_i \neq -1$ and $\beta_{\omega_1, \dots, \omega_i} \neq 0$, hence $\omega_1 + \cdots + \omega_i \geq -1$. Therefore $\check{\omega}_i \geq 0$ and $\hat{\omega}_{i+1} = \|\omega\| - \check{\omega}_i \leq \|\omega\|$, while $\hat{\omega}_1 = \|\omega\|$. \square

8.4 We recall that the convergent series $a_\eta(x)$ were defined in (3.2) as Taylor coefficients with respect to y of the saddle-node vector field (2.1). We define $\tilde{\varphi}_n(z), \tilde{\psi}_n(z), \tilde{a}_\eta(z), \tilde{\mathcal{V}}^\omega(z), \tilde{\mathcal{V}}^{\omega}(z)$ from $\varphi_n(x), \psi_n(x), a_\eta(x), \mathcal{V}^\omega(x), \mathcal{V}^{\omega}(x)$ by the change of variable $z = -1/x$ (for any $n \in \mathbb{N}, \eta \in \Omega, \omega \in \Omega^\bullet$), and we denote by $\widehat{\varphi}_n(\zeta), \widehat{\psi}_n(\zeta)$, etc. the formal Borel transforms of these formal series.

In view of Lemma 3.2, the formal series $\tilde{\mathcal{V}}^\omega$ are uniquely determined by the equations $\tilde{\mathcal{V}}^\emptyset = 1$ and

$$\left(\frac{d}{dz} + \|\omega\| \right) \tilde{\mathcal{V}}^\omega = \tilde{a}_{\omega_1} \tilde{\mathcal{V}}^{\omega'}, \quad \tilde{\mathcal{V}}^\omega \in z^{-1} \mathbb{C}[[z^{-1}]]$$

for ω non-empty, with ω' denoting ω deprived from its first letter. Since \mathcal{B} transforms $\frac{d}{dz}$ into multiplication by $-\zeta$ and multiplication into convolution, we get $\widehat{\mathcal{V}}^\emptyset = \delta$ and

$$\widehat{\mathcal{V}}^\omega(\zeta) = -\frac{1}{\zeta - \|\omega\|} (\widehat{a}_{\omega_1} * \widehat{\mathcal{V}}^{\omega'}), \quad \omega \neq \emptyset,$$

where the right-hand side belongs to $\mathbb{C}[[\zeta]]$ even if $\|\omega\| = 0$, by the same argument as in the proof of Lemma 3.2. It belongs in fact to $\mathbb{C}\{\zeta\}$, by induction on $r(\omega)$, and

$$\widehat{\mathcal{V}}^\omega = (-1)^r \frac{1}{\zeta - \hat{\omega}_1} \left(\widehat{a}_{\omega_1} * \left(\frac{1}{\zeta - \hat{\omega}_2} \left(\widehat{a}_{\omega_2} * \left(\cdots \left(\frac{1}{\zeta - \hat{\omega}_r} \widehat{a}_{\omega_r} \right) \cdots \right) \right) \right) \right) \quad (8.9)$$

with the notation of (8.7). In view of the stability properties of $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ (stability by convolution with another element of $\widehat{H}(\mathcal{R}(\mathbb{Z}))$, a fortiori with an entire function, or by multiplication with a meromorphic function regular on $\mathbb{C} \setminus \mathbb{Z}^*$), this implies that the functions $\widehat{\mathcal{V}}^\omega$ are resurgent, as announced in the introduction to this section. We shall give more details on this later.

8.5 Here is a first consequence for the functions $\widehat{\varphi}_n$ and $\widehat{\psi}_n$:

Lemma 8.2. *For each $n \in \mathbb{N}$,*

$$\widehat{\varphi}_n = \sum_{\|\omega\|=n-1} \beta_\omega \widehat{\mathcal{V}}^\omega, \quad \widehat{\psi}_n = \sum_{\|\omega\|=n-1} \beta_\omega \widehat{\mathcal{V}}^\omega, \quad (8.10)$$

with formally convergent series in $\mathbb{C}[[\zeta]]$, and for each non-empty ω such that $\|\omega\| = n - 1$,

$$\beta_\omega \widehat{\mathcal{V}}^\omega = \mathcal{S}_{\hat{\omega}_1} \mathcal{A}_{\omega_1} \cdots \mathcal{S}_{\hat{\omega}_r} \mathcal{A}_{\omega_r} \delta, \quad \beta_\omega \widehat{\mathcal{V}}^\omega = \frac{1}{\zeta - (n-1)} \mathcal{A}_{\omega_r} \mathcal{S}_{\check{\omega}_{r-1}} \mathcal{A}_{\omega_{r-1}} \cdots \mathcal{S}_{\check{\omega}_1} \mathcal{A}_{\omega_1} \delta, \quad (8.11)$$

with convolution operators

$$\mathcal{A}_\eta: \widehat{\varphi} \mapsto \widehat{a}_\eta * \widehat{\varphi}, \quad \eta \in \Omega$$

and multiplication operators

$$\mathcal{S}_m: \widehat{\varphi} \mapsto -\frac{n-m}{\zeta - m} \widehat{\varphi}, \quad \mathcal{S}_m: \widehat{\varphi} \mapsto \frac{m+1}{\zeta - m} \widehat{\varphi}, \quad m \in \mathbb{Z} \quad (8.12)$$

Proof. Formula (8.10) is a direct consequence of (8.6).

In order to deal with $\widehat{\mathcal{V}}^\omega$, we pass from $\omega = (\omega_1, \dots, \omega_r)$ to $\tilde{\omega} = (\omega_r, \dots, \omega_1)$ and this exchanges $\hat{\omega}_i$ and $\check{\omega}_{r-i+1}$, thus (8.9) implies

$$\widehat{\mathcal{V}}^\omega = \frac{1}{\zeta - \check{\omega}_r} \left(\widehat{a}_{\omega_r} * \left(\frac{1}{\zeta - \check{\omega}_{r-1}} \left(\widehat{a}_{\omega_{r-1}} * \left(\cdots \left(\frac{1}{\zeta - \check{\omega}_1} \widehat{a}_{\omega_1} \right) \cdots \right) \right) \right) \right).$$

Since $\check{\omega}_r = n - 1$, multiplying by $\beta_\omega = (\check{\omega}_{r-1} + 1) \cdots (\check{\omega}_1 + 1)$, we get the second part of (8.11). The first part of this formula is obtained by multiplying (8.9) by β_ω written in the form $\beta_\omega = (n - \hat{\omega}_1)(n - \hat{\omega}_2) \cdots (n - \hat{\omega}_r)$ (indeed, $n - \hat{\omega}_1 = 1$ and $n - \hat{\omega}_i = \check{\omega}_{i-1} + 1$ for $2 \leq i \leq r$). \square

8.6 The appearance of singularities in our problem is due to the multiplication operators $\mathcal{S}_{\hat{\omega}_i}$ or $\mathcal{S}_{\check{\omega}_i}$. In view of Lemma 8.1 and formulas (8.11)–(8.12), we are led to introduce subspaces of $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ formed of functions with smaller sets of singularities. We do this by considering Riemann surfaces $\mathcal{R}(\mathcal{P})$ slightly more general than $\mathcal{R}(\mathbb{Z})$.

Let \mathcal{P} denote a subset of \mathbb{Z} . We define the Riemann surface $\mathcal{R}(\mathcal{P})$ as the set of all homotopy classes of rectifiable oriented paths which start from the origin and then avoid \mathcal{P} . The Riemann surface $\mathcal{R}(\mathcal{P})$ and the universal cover of $\mathbb{C} \setminus \mathcal{P}$ coincide if $0 \notin \mathcal{P}$; there is a difference between them when $0 \in \mathcal{P}$: there is no point which projects onto 0 in the second one, while the first one still has an “origin”.

The space $\widehat{H}(\mathcal{R}(\mathcal{P}))$ of all holomorphic functions of $\mathcal{R}(\mathcal{P})$ can be identified with the space of all $\widehat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ which admit an analytic continuation along any representative of any element of $\mathcal{R}(\mathcal{P})$. It can thus also be identified with the subspace of $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ consisting of those functions holomorphic in $\mathcal{R}(\mathbb{Z})$, the branches of which are regular at each point of $\mathbb{Z} \setminus \mathcal{P}$.

We shall particularly be interested in two cases: $\mathcal{P}^- = n - \mathbb{N}^*$ and $\mathcal{P}^+ = \mathbb{N}$. Indeed, our aim is to show that the functions $\widehat{\varphi}_n$ belong to $\widehat{H}(\mathcal{R}(n - \mathbb{N}^*))$ for

any $n \in \mathbb{N}$ and that the functions $\widehat{\psi}_n$ belong to $\widehat{H}(\mathcal{R}(\mathbb{N}))$ for any $n \geq 1$, while $(\zeta + 1)\widehat{\psi}_1(\zeta) \in \widehat{H}(\mathcal{R}(\mathbb{N}))$.

One could prove (with the help of symmetrically contractile paths) that the spaces $\widehat{H}(\mathcal{R}(\mathbb{N}))$, $\widehat{H}(\mathcal{R}(-\mathbb{N}^*))$ or $\widehat{H}(\mathcal{R}(-\mathbb{N}))$ are stable by convolution because the corresponding sets \mathcal{P} are stable by addition, but beware that this is not the case of $\widehat{H}(\mathcal{R}(n - \mathbb{N}^*))$ if $n \geq 2$.

8.7 As previously mentioned, for each $\omega \neq \emptyset$, $\beta_\omega \widehat{\mathcal{V}}^\omega$ and $\beta_\omega \widehat{\mathcal{V}}^\omega$ belong to $\widehat{H}(\mathcal{R}(\mathbb{Z}))$ by virtue of general stability properties. But formula (8.11) permits a more elementary argument and more precise conclusions.

Indeed $\mathcal{A}_{\omega_r} \delta = \widehat{a}_{\omega_r}$, resp. $\mathcal{A}_{\omega_1} \delta = \widehat{a}_{\omega_1}$, is an entire function, which vanishes at the origin if $\omega_r = 0$, resp. $\omega_1 = 0$. Thus

$$\mathcal{S}_{\widehat{\omega}_r} \mathcal{A}_{\omega_r} \delta = -\frac{n - \omega_r}{\zeta - \omega_r} \widehat{a}_{\omega_r}, \quad \text{resp. } \mathcal{S}_{\widehat{\omega}_1} \mathcal{A}_{\omega_1} \delta = \frac{\omega_1 + 1}{\zeta - \omega_1} \widehat{a}_{\omega_1}, \quad (8.13)$$

is meromorphic on \mathbb{C} and regular at the origin if $\omega_r \neq 0$, resp. $\omega_1 \neq 0$, and entire if $\omega_r = 0$, resp. $\omega_1 = 0$. In fact,

$$\widehat{\omega}_r \leq n - 1 \Rightarrow \mathcal{S}_{\widehat{\omega}_r} \mathcal{A}_{\omega_r} \delta \in \widehat{H}(\mathcal{R}(n - \mathbb{N}^*)), \quad \check{\omega}_1 \geq 0 \Rightarrow \mathcal{S}_{\widehat{\omega}_1} \mathcal{A}_{\omega_1} \delta \in \widehat{H}(\mathcal{R}(\mathbb{N})).$$

Therefore, one can apply $r - 1$ times the following

Lemma 8.3. Suppose that $\mathcal{P} \subset \mathbb{Z}$, $\widehat{\varphi} \in \widehat{H}(\mathcal{R}(\mathcal{P}))$ and \widehat{b} is entire. Then $\widehat{b} * \widehat{\varphi} \in \widehat{H}(\mathcal{R}(\mathcal{P}))$. If furthermore \widehat{s} is a meromorphic function, the poles of which all belong to \mathcal{P} and with at most a simple pole at the origin, then $\widehat{s}(\widehat{b} * \widehat{\varphi}) \in \widehat{H}(\mathcal{R}(\mathcal{P}))$.

Consider a rectifiable oriented path with arc-length parametrisation $\gamma: [0, T] \rightarrow \mathbb{C}$, such that $\gamma(0) = 0$ and $\gamma(t) \in \mathbb{C} \setminus \mathcal{P}$ for $0 < t \leq T$. Denoting the analytic continuation of $\widehat{\varphi}$ along γ by the same symbol $\widehat{\varphi}$, we suppose moreover that

$$|\widehat{\varphi}(\gamma(t))| \leq P(t) e^{Ct}, \quad 0 \leq t \leq T,$$

with a continuous function P and a constant $C \geq 0$, and that there is a continuous monotonic non-decreasing function Q such that $|\widehat{b}(\zeta)| \leq Q(|\zeta|) e^{C|\zeta|}$ for all $\zeta \in \mathbb{C}$. Then, for all $t \in [0, T]$, the analytic continuation of $\widehat{b} * \widehat{\varphi}$ at $\zeta = \gamma(t)$ satisfies

$$\widehat{b} * \widehat{\varphi}(\zeta) = \int_{\gamma_\zeta} \widehat{b}(\zeta - \zeta') \widehat{\varphi}(\zeta') d\zeta', \quad |\widehat{b} * \widehat{\varphi}(\gamma(t))| \leq P * Q(t) e^{Ct}, \quad (8.14)$$

with γ_ζ denoting the restriction $\gamma|_{[0,t]}$ and $P * Q(t) = \int_0^t P(t') Q(t - t') dt'$.

Proof. The first statement and the first part of (8.14) are obtained by means of the Cauchy theorem: if γ_1 and γ_2 are two representatives of the same element of $\mathcal{R}(\mathcal{P})$ and $\xi = \pi([\gamma_1]) = \pi([\gamma_2])$, then $\int_{\gamma_1} \widehat{b}(\xi - \xi') \widehat{\varphi}(\xi') d\xi'$ and $\int_{\gamma_2} \widehat{b}(\xi - \xi') \widehat{\varphi}(\xi') d\xi'$ coincide; one can check that the function thus defined on $\mathcal{R}(\mathcal{P})$ is holomorphic and this is clearly an extension of $\widehat{b} * \widehat{\varphi}$. Moreover $\widehat{b} * \widehat{\varphi}$ vanishes at the origin, thus $\widehat{s}(\widehat{b} * \widehat{\varphi}) \in \widehat{H}(\mathcal{R}(\mathcal{P}))$ even if \widehat{s} has a simple pole at 0.

We thus have

$$\widehat{b} * \widehat{\varphi}(\gamma(t)) = \int_0^t \widehat{b}(\gamma(t) - \gamma(t')) \widehat{\varphi}(\gamma(t')) \dot{\gamma}(t') dt'.$$

For almost every $t' \in [0, t]$, $|\dot{\gamma}(t')| = 1$ and $|\gamma(t) - \gamma(t')| \leq t - t'$, whence $|\widehat{b}(\gamma(t) - \gamma(t'))| \leq Q(t - t') e^{C(t-t')}$ by monotonicity of $\xi \mapsto Q(\xi) e^\xi$. The conclusion follows. \square

In view of Lemma 8.1 and formula (8.11), the first part of Lemma 8.3 implies

Corollary 8.1. *Let $n \in \mathbb{N}$ and ω be a non-empty word such that $\|\omega\| = n - 1$. Then the function $\beta_\omega \widehat{\mathcal{V}}^\omega$ belongs to $\widehat{H}(\mathcal{R}(n - \mathbb{N}^*))$ and the function*

$$\zeta \mapsto (\zeta - (n - 1)) \beta_\omega \widehat{\mathcal{V}}^\omega(\zeta)$$

belongs to $\widehat{H}(\mathcal{R}(\mathbb{N}))$.

8.8 Our aim is now to exploit formula (8.11) and the quantitative information contained in Lemma 8.3 to produce upper bounds for

$$|\beta_\omega \widehat{\mathcal{V}}^\omega(\zeta)|, \quad \text{resp. } |(\zeta - (n - 1)) \beta_\omega \widehat{\mathcal{V}}^\omega(\zeta)|$$

which will ensure the uniform convergence of the series (8.10) (up to the factor $\zeta - (n - 1)$ for the second one) in any compact subset of $\mathcal{R}(n - \mathbb{N}^*)$, resp. $\mathcal{R}(\mathbb{N})$.

We first choose positive constants K, L, C such that

$$|\widehat{a}_\eta(\zeta)| \leq KL^\eta e^{C|\zeta|}, \quad \zeta \in \mathbb{C}, \eta \in \Omega. \quad (8.15)$$

This is possible, since $\sum \frac{a_\eta(x)}{x} y^{\eta+1} = \frac{A(x, y) - y}{y} \in \mathbb{C}\{x, y\}$ by assumption, thus one can find constants such that $|\frac{a_\eta(x)}{x}| \leq KL^\eta$ for $|x| \leq C^{-1}$ and use (8.1). We can also assume, possibly at the price of increasing of K , that

$$|\widehat{a}_0(\zeta)| \leq K|\zeta| e^{C|\zeta|}, \quad \zeta \in \mathbb{C}, \quad (8.16)$$

since $a_0(x) \in x^2 \mathbb{C}\{x\}$.

8.9 Next, we define exhaustions of $\mathcal{R}(n - \mathbb{N}^*)$, resp. $\mathcal{R}(\mathbb{N})$, by subsets $\mathcal{R}_{\rho, N}(n - \mathbb{N}^*)$, resp. $\mathcal{R}_{\rho, N}(\mathbb{N})$, in which we shall be able to derive appropriate bounds for our functions. Let $\rho \in]0, \frac{1}{2}[$ and $N \in \mathbb{N}^*$.

We denote by $\dot{\mathcal{R}}_{\rho, N}(n - \mathbb{N}^*)$ the subset of \mathbb{C} obtained by removing the open discs $D(m, \rho)$ with radius ρ and integer centres $m \leq n - 1$, and removing also the points ζ such that the segment $[0, \zeta]$ intersect the open disc $D(-N, \rho)$ (i.e. the points which are hidden by $D(-N, \rho)$ to an observer located at the origin).

Similarly, we denote by $\dot{\mathcal{R}}_{\rho, N}(\mathbb{N})$ the subset of \mathbb{C} obtained by removing the open discs $D(m, \rho)$ with radius ρ and integer centres $m \geq 0$, and removing also the points ζ such that the segment $[0, \zeta]$ intersect the open disc $D(N, \rho)$. Thus,

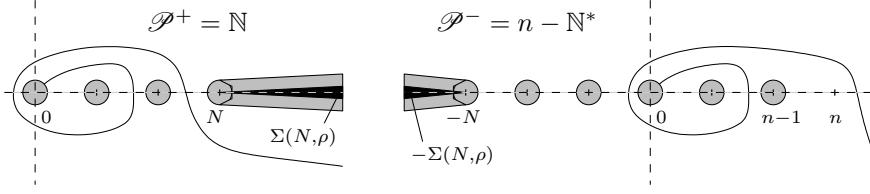


Figure 1. The set $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}^\pm)$ and the image of a $(\rho, N, \mathcal{P}^\pm)$ -adapted path.

with the notations $\mathcal{P}^- = n - \mathbb{N}^*$ and $\mathcal{P}^+ = \mathbb{N}$,

$$\begin{aligned}\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}^\pm) &= \{ \zeta \in \mathbb{C} \mid \text{dist}(\zeta, \mathcal{P}^\pm) \geq \rho \text{ and } \text{dist}(\pm N, [0, \zeta]) \geq \rho \} \\ &= \{ \zeta \in \mathbb{C} \mid \text{dist}(\zeta, \mathcal{P}^\pm \cup \pm \Sigma(\rho, N)) \geq \rho \},\end{aligned}$$

with the notation Σ introduced after the statement of Theorem 2; see Figure 1.

Now, for $\mathcal{P} = \mathcal{P}^\pm$, consider the rectifiable oriented paths γ which start at the origin and either stay in the disc $D(0, \rho)$, or leave it and then stay in $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P})$. The homotopy classes of such paths form a set $\mathcal{R}_{\rho,N}(\mathcal{P})$ which we can identify with a subset of $\mathcal{R}(\mathcal{P})$.

Definition 8.2. If the arc-length parametrisation of a rectifiable oriented path $\gamma: [0, T] \rightarrow \mathbb{C}$ satisfies, for each $t \in [0, T]$,

$$\begin{aligned}0 \leq t \leq \rho &\Rightarrow |\gamma(t)| = t, \\ t > \rho &\Rightarrow \gamma(t) \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}),\end{aligned}$$

then we say that the parametrised path γ is (ρ, N, \mathcal{P}) -adapted. We speak of infinite (ρ, N, \mathcal{P}) -adapted path if γ is defined on $[0, +\infty[$.

One can characterize $\mathcal{R}_{\rho,N}(\mathcal{P})$ as follows: a point of $\mathcal{R}(\mathcal{P})$ belongs to $\mathcal{R}_{\rho,N}(\mathcal{P})$ iff it can be represented by a (ρ, N, \mathcal{P}) -adapted path.

Observe that the projection onto \mathbb{C} of $\mathcal{R}_{\rho,N}(\mathcal{P})$ is $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0, \rho)$ (only for $\mathcal{P} = -\mathbb{N}^*$ is $D(0, \rho)$ contained in $\dot{\mathcal{R}}_{\rho,N}(\mathcal{P})$) and that $\mathcal{R}(\mathcal{P}) = \bigcup_{\rho, N} \mathcal{R}_{\rho,N}(\mathcal{P})$.

8.10 We now show how to control the operators S_m and \mathfrak{S}_m uniformly in a set $\mathcal{R}_{\rho,N}(\mathcal{P}^\pm)$:

Lemma 8.4. Let $n \in \mathbb{N}$ and S_m, \mathfrak{S}_m as in (8.12), and consider the meromorphic functions $S_m = S_m 1$ and $\mathfrak{S}_m = \mathfrak{S}_m 1$.

Given ρ, N as above, there exist $\lambda > 0$ which depends only on ρ, N and $\lambda_n > 0$ which depends only on ρ, N, n such that, for $m \in \mathcal{P} \setminus \{0\}$,

$$\text{if } \mathcal{P} = n - \mathbb{N}^*: \quad |S_m(\zeta)| \leq \lambda_n \quad \text{for } \zeta \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0, \rho) \quad (8.17)$$

$$\text{if } \mathcal{P} = \mathbb{N}: \quad |S_m(\zeta)| \leq \lambda \quad \text{for } \zeta \in \dot{\mathcal{R}}_{\rho,N}(\mathcal{P}) \cup D(0, \rho) \quad (8.17')$$

and

$$|S_0(\zeta)| \leq \lambda_n, \quad |S_0(\zeta)| \leq \lambda, \quad \text{for } \zeta \in \mathbb{C} \setminus D(0, \rho) \quad (8.18)$$

$$|S_0(\zeta)| \leq \frac{\rho \lambda_n}{|\zeta|}, \quad |S_0(\zeta)| \leq \frac{\rho \lambda}{|\zeta|}, \quad \text{for } \zeta \in D(0, \rho). \quad (8.19)$$

One can take $\lambda = (N+1)\rho^{-1}$ and $\lambda_n = (|n|+N)\rho^{-1}$.

Proof. Let $m \in \mathcal{P} \setminus \{0\}$ and $\zeta \in \dot{\mathcal{R}}_{\rho, N}(\mathcal{P}) \cup D(0, \rho)$, thus $|\zeta - m| \geq \rho$.

Consider first the case $\mathcal{P} = \mathbb{N}$. If $m \geq N$, then $|\zeta - m| \geq \frac{\rho|m|}{N}$ by Thales theorem; thus $\frac{1}{|\zeta-m|} \leq \rho^{-1}$ and $|\frac{m}{\zeta-m}| \leq N\rho^{-1}$ for any $m \in \mathbb{N}^*$. Therefore $|S_m(\zeta)| = \left| \frac{m+1}{\zeta-m} \right| \leq \lambda = (N+1)\rho^{-1}$. Since $\lambda \geq \rho^{-1}$, $S_0(\zeta) = 1/\zeta$ also satisfies the required inequalities.

When $\mathcal{P} = n - \mathbb{N}^*$, one argues similarly except that the case $N \leq m \leq n-1$ must be treated separately. \square

8.11 Combining the previous two lemmas, we get

Lemma 8.5. *Let us fix n, ρ, N as above, K, L, C as in (8.15)–(8.16) and λ, λ_n as in Lemma 8.4. Suppose that $\mathcal{P} = n - \mathbb{N}^*$ or \mathbb{N} , $\gamma: [0, T] \rightarrow \mathbb{C}$ is (ρ, N, \mathcal{P}) -adapted and $\hat{\varphi} \in \hat{H}(\mathcal{R}(\mathcal{P}))$ satisfies*

$$|\hat{\varphi}(\gamma(t))| \leq P(t) e^{Ct}, \quad 0 \leq t \leq T,$$

with a continuous monotonic non-decreasing function P and a constant $C \geq 0$. Assume $m \in \mathcal{P}$, with the restriction $m \neq 0$ if $n = 0$ and $\mathcal{P} = -\mathbb{N}^*$.

Then, for any $\eta \in \Omega$,

$$\mathcal{P} = n - \mathbb{N}^* \Rightarrow \mathcal{S}_m \mathcal{A}_\eta \hat{\varphi} \in \hat{H}(\mathcal{R}(n - \mathbb{N}^*)), \quad \mathcal{P} = \mathbb{N} \Rightarrow \mathcal{S}_m \mathcal{A}_\eta \hat{\varphi} \in \hat{H}(\mathcal{R}(\mathbb{N})),$$

and, in the first case,

$$m \neq 0 \text{ or } \eta = 0 \Rightarrow |\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi}(\gamma(t))| \leq \lambda_n K L^\eta (1 * P)(t) e^{Ct} \quad (8.20)$$

$$m = 0 \text{ and } \eta \neq 0 \Rightarrow |\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi}(\gamma(t))| \leq \lambda_n K L^\eta ((\delta + 1) * P)(t) e^{Ct} \quad (8.21)$$

for all $t \in [0, T]$, while in the second case the function $\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi}$ satisfies the same inequalities with λ replacing λ_n .

Proof. We suppose $\mathcal{P} = n - \mathbb{N}^*$ and show the properties for $\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi}$ only, the other case being similar. Since $\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi} = S_m(\hat{a}_\eta * \hat{\varphi})$, this function belongs to $\hat{H}(\mathcal{R}(\mathcal{P}))$ by the first part of Lemma 8.3. In view of (8.15)–(8.16), the second part of this lemma yields

$$|\mathcal{A}_\eta \hat{\varphi}(\gamma(t))| \leq K L^\eta (1 * P)(t) e^{Ct} \quad (8.22)$$

$$|\mathcal{A}_0 \hat{\varphi}(\gamma(t))| \leq K (I * P)(t) e^{Ct} \quad (8.23)$$

for all $t \in [0, T]$, with $I(t) \equiv t$ (notice that the first inequality holds if $\eta = 0$ as well).

If $m \neq 0$, then (8.17) yields the desired inequality for $|\mathcal{S}_m \mathcal{A}_\eta \hat{\varphi}(\gamma(t))|$.

Suppose $m = 0$; thus $n \neq 0$ by assumption. We observe that, if $t > \rho$, then $\gamma(t) \in \dot{\mathcal{R}}_{\rho, N}(\mathcal{P})$ has modulus $> \rho$ and (8.18) yields $|S_0(\gamma(t))| \leq \lambda_n$, whereas if $t \leq \rho$, then $|\gamma(t)| = |t|$ and (8.19) yields $|S_0(\gamma(t))| \leq \frac{\rho \lambda_n}{t}$.

Thus, if $m = 0$ and $\eta = 0$, then (8.22) yields the desired inequality when $t > \rho$ and (8.23) yields $|\mathcal{S}_0 \mathcal{A}_0 \widehat{\varphi}(\gamma(t))| \leq K \rho \lambda_n \frac{I^* P(t)}{t} e^{Ct}$ for $t \leq \rho$, which is sufficient since $\frac{I^* P(t)}{t} = \frac{1}{t} \int_0^t t' P(t-t') dt' \leq 1 * P(t)$ and $\rho < 1$.

We conclude with the case where $m = 0$ and $\eta \neq 0$. Using (8.22), we obtain the result when $t > \rho$, since $1 * P \leq P + 1 * P$. When $t \leq \rho$, we get $|\mathcal{S}_0 \mathcal{A}_\eta \widehat{\varphi}(\gamma(t))| \leq K L^\eta \rho \lambda_n \frac{1 * P(t)}{t} e^{Ct}$, which is sufficient since $\frac{1 * P(t)}{t} = \frac{1}{t} \int_0^t P(t') dt' \leq P(t)$. \square

8.12 End of the proof of Theorem 2: case of $\widehat{\varphi}_n$.

Let $n \in \mathbb{N}$. According to (8.10), the formal series $\widehat{\varphi}_n$ can be written as the formally convergent series $\sum_{\|\omega\|=n-1} \beta_\omega \widehat{\mathcal{V}}^\omega$. Let $\mathcal{P} = n - \mathbb{N}^*$; according to Corollary 8.1 each $\beta_\omega \widehat{\mathcal{V}}^\omega$ converges to a function of $\widehat{H}(\mathcal{R}(\mathcal{P}))$, it is thus sufficient to check the uniform convergence of the above series *as a series of holomorphic functions* in each compact subset of $\mathcal{R}(\mathcal{P})$ and to give appropriate bounds. Let us fix $\rho \in]0, \frac{1}{2}[$, $N \in \mathbb{N}^*$ and $K, L, C, \lambda, \lambda_n$ as in Lemma 8.5.

– We first show that, for any (ρ, N, \mathcal{P}) -adapted path γ (infinite or not) and for any $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$ with $r \geq 1$ and $\|\omega\| = n-1$, one has for all t

$$|\beta_\omega \widehat{\mathcal{V}}^\omega(\gamma(t))| \leq (\lambda_n K)^r L^{n-1} \widehat{P}_r(t) e^{Ct}, \quad \widehat{P}_r = (\delta+1)^{* \lfloor r/2 \rfloor} * 1^{*\lceil r/2 \rceil}, \quad (8.24)$$

with the same notation as in (3.8) for $\lceil r/2 \rceil$ and $\lfloor r/2 \rfloor = r - \lceil r/2 \rceil$. Observe that \widehat{P}_r is a polynomial with non-negative coefficients.

If $\omega_r = \dot{\omega}_r \neq 0$, then (8.15) and (8.17) yield $|\mathcal{S}_{\dot{\omega}_r} \widehat{a}_{\omega_r}(\zeta)| \leq \lambda_n K L^{\omega_r} e^{C|\zeta|}$ for all $\zeta \in \mathcal{R}(\mathcal{P})$. The same inequality holds also if $\omega_r = 0$ (use (8.15) and (8.18) if $|\zeta| > \rho$, and (8.16) and (8.19) if $|\zeta| \leq \rho$). Therefore

$$|\mathcal{S}_{\dot{\omega}_r} \widehat{a}_{\omega_r}(\gamma(t))| \leq \lambda_n K L^{\omega_r} e^{Ct}, \quad t \geq 0 \quad (8.25)$$

(since $|\gamma(t)| \leq t$). Since Lemma 8.1 implies $\dot{\omega}_1, \dots, \dot{\omega}_{r-1} \leq n-1$, we can apply $r-1$ times Lemma 8.5 and get

$$|\beta_\omega \widehat{\mathcal{V}}^\omega(\gamma(t))| \leq (\lambda_n K)^r L^{n-1} ((\delta+1)^{* (r-a)} * 1^{*a})(t) e^{Ct}, \quad (8.26)$$

with $a = \text{card}\{ i \in [1, r] \mid \dot{\omega}_i \neq 0 \text{ or } \omega_i = 0 \}$. But $a \geq \lceil r/2 \rceil$, as was shown in Lemma 3.2, hence the polynomial expression in t appearing in the right-hand side of (8.26) can be written $(\delta+1)^{* (r-a)} * 1^{*(a-\lceil r/2 \rceil)} * 1^{*\lceil r/2 \rceil} \leq (\delta+1)^{* (r-\lceil r/2 \rceil)} * 1^{*\lceil r/2 \rceil}$, which yields (8.24).

– We have

$$\text{card}\{ \omega \in \Omega^r \mid \|\omega\| = n-1 \} = \text{card}\{ k \in \mathbb{N}^r \mid \|k\| = n+r-1 \}$$

$$= \binom{n+2(r-1)}{r-1} \leq 2^{n+2(r-1)},$$

hence, for each $r \geq 1$,

$$\sum_{r(\omega)=r, \|\omega\|=n-1} |\beta_\omega \hat{\mathcal{V}}^\omega(\gamma(t))| \leq 2\lambda_n K(2L)^{n-1} \Lambda_n^{r-1} \hat{P}_r(t) e^{Ct} \quad (8.27)$$

with $\Lambda_n = 4\lambda_n K$. But $\tilde{P}_r(z) = \mathcal{B}^{-1} \hat{P}_r = (1+z^{-1})^{\lfloor r/2 \rfloor} z^{-\lceil r/2 \rceil}$ gives rise to

$$\tilde{\Phi}_n(z) = \sum_{r \geq 1} \Lambda_n^{r-1} \tilde{P}_r(z) = z^{-1} (1 + \Lambda_n(1+z^{-1})) (1 - \Lambda_n^2(z^{-1} + z^{-2}))^{-1}$$

which is convergent (with non-negative coefficients), thus $\sum_{r \geq 1} \Lambda_n^{r-1} \hat{P}_r(t) = \mathcal{B} \tilde{\Phi}_n(t)$

is convergent for all t . Therefore $\hat{\varphi}_n$ is the sum of a series of holomorphic functions uniformly convergent in every compact subset of $\mathcal{R}_{\rho, N}(\mathcal{P})$ satisfying

$$|\hat{\varphi}_n(\gamma(t))| \leq 2\lambda_n K(2L)^{n-1} \mathcal{B} \tilde{\Phi}_n(t) e^{Ct}.$$

– We conclude by using inequalities of the form (8.1) to bound $\mathcal{B} \tilde{\Phi}_n$: one can check that $|z| \geq 4\Lambda_n^2$ implies $|z \tilde{\Phi}_n(z)| \leq 2(2 + \Lambda_n)$, hence

$$\mathcal{B} \tilde{\Phi}_n(t) \leq 2(2 + \Lambda_n) e^{4\Lambda_n^2 t}.$$

In view of the explicit dependence of λ_n on n indicated in Lemma 8.4, we easily get inequalities of the form (8.3) (possibly with larger constants K, L, C).

8.13 End of the proof of Theorem 2: case of $\hat{\psi}_n$.

We only indicate the inequalities that one obtains when adapting the previous arguments to the case of $\hat{\psi}_n$. Let $\mathcal{P} = \mathbb{N}$ and

$$\hat{\chi}_n(\zeta) = (\zeta - (n-1)) \hat{\psi}_n(\zeta), \quad \hat{\mathcal{W}}^\omega(\zeta) = (\zeta - (n-1)) \beta_\omega \hat{\mathcal{V}}^\omega(\zeta).$$

The initial bound corresponding to (8.25) is $|\mathcal{S}_{\omega_1} \hat{a}_{\omega_1}(\gamma(t))| \leq \lambda K L^{\omega_1} e^{Ct}$. This yields

$$|\hat{\mathcal{W}}^\omega(\gamma(t))| \leq K(\lambda K)^{r-1} L^{n-1} \hat{Q}_r(t) e^{Ct}, \quad \hat{Q}_r = (\delta+1)^{*[\frac{r}{2}-1]} * 1^{*\lceil \frac{r}{2}+1 \rceil} \quad (8.28)$$

after $r-2$ applications of Lemma 8.5 (with an intermediary inequality analogous to (8.26) but involving $b = 1 + \text{card}\{i \in [1, r-1] \mid \check{\omega}_i \neq 0 \text{ or } \omega_i = 0\} \geq \lfloor \frac{r}{2} + 1 \rfloor$ instead of a).

Therefore $|\hat{\chi}_n(\gamma(t))| \leq 2K(2L)^{n-1} \mathcal{B} \tilde{\Psi}(t) e^{Ct}$, with

$$\tilde{\Psi}(z) = \sum_{r \geq 1} \Lambda^{r-1} \mathcal{B}^{-1} \hat{Q}_r(z) = z^{-1} (1 + \Lambda z^{-1}) (1 - \Lambda^2(z^{-1} + z^{-2}))^{-1}, \quad \Lambda = 4\lambda K,$$

whence $|\hat{\chi}_n(\gamma(t))| \leq K_1(2L)^n e^{C_1 t}$, with suitable constants K_1, C_1 independent of n .

This is the desired conclusion when $n = 0$. When $n \geq 1$, we can pass from $\hat{\chi}_n$ to $\hat{\psi}_n$ since $|\gamma(t) - (n-1)| \geq \rho$, with only one exception; namely, if $n = 1$ and $t < \rho$, then we only have a bound for $|\zeta \hat{\psi}_1(\zeta)|$ with $\zeta = \gamma(t) \in D(0, \rho)$, but in that case the analyticity of $\hat{\chi}_1$ at the origin of $\mathcal{R}(\mathbb{N})$ is sufficient since we know that its Taylor series has no constant term.

8.14 Proof of inequalities (8.5). They will follow from a lemma which has its own interest.

Lemma 8.6. *For every $n \in \mathbb{N}$, the following identity holds in $\mathbb{C}[[x]]$:*

$$\varphi_n = \sum_{\substack{s \geq 1, n_1, \dots, n_s \geq 0 \\ n_1 + \dots + n_s = n+s-1}} \frac{(-1)^s}{s} \binom{n+s-1}{s-1} \psi_{n_1} \cdots \psi_{n_s}, \quad (8.29)$$

where the right-hand side is a formally convergent series.

Proof. This is the consequence of the following version of Lagrange inversion formula: If $\chi(t, y) \in \mathbb{C}[[t, y]]$, then the formal transformation

$$(t, x, y) \mapsto (t, x, y - x\chi(t, y))$$

has an inverse of the form $(t, x, y) \mapsto (t, x, \mathcal{Y}(t, x, y))$ with $\mathcal{Y} \in \mathbb{C}[[t, x, y]]$ given by

$$\mathcal{Y}(t, x, y) = y + \sum_{s \geq 1} \frac{x^s}{s!} \left(\frac{\partial}{\partial y} \right)^{s-1} (\chi(t, y)^s). \quad (8.30)$$

(Proof: The transformation is invertible, because its 1-jet is, and the inverse must be of the form $(t, x, \mathcal{Y}(t, x, y))$ with $\mathcal{Y}(t, 0, y) = y$ and $\partial_y \mathcal{Y}(t, 0, y) = 1$. It is thus sufficient to check the formula

$$\partial_x^s \mathcal{Y}(t, x, y) = \left(\frac{\partial}{\partial y} \right)^{s-1} \left[(\chi(t, \mathcal{Y}(t, x, y)))^s \partial_y \mathcal{Y}(t, x, y) \right], \quad s \geq 1$$

by induction on s , which is easy.)

Since $\psi_n(x) \in x\mathbb{C}[[x]]$, we can apply this with $\chi(t, y) = \sum_{n \geq 0} \chi_n(t)y^n$ where $\chi_n(x) = -\frac{\psi_n(x)}{x}$: this way $y - x\chi(x, y) = y + \sum_{n \geq 0} \psi_n(x)y^n = \psi(x, y)$, and (8.30) yields

$$\varphi(x, y) = y + \sum_{n \geq 0} \varphi_n(x)y^n = \sum_{s \geq 1} \frac{(-1)^s}{s!} \left(\frac{\partial}{\partial y} \right)^{s-1} \left[\left(\sum_{n \geq 0} \psi_n(x)y^n \right)^s \right]$$

by specialization to $t = x$, whence the result follows (one gets a formally convergent series because $\psi_n(x) \in x\mathbb{C}[[x]]$). \square

As a consequence, we get

$$\widehat{\varphi}_n = \sum_{\substack{s \geq 1, n_1, \dots, n_s \geq 0 \\ n_1 + \dots + n_s = n+s-1}} \frac{(-1)^s}{s} \binom{n+s-1}{s-1} \widehat{\psi}_{n_1} * \cdots * \widehat{\psi}_{n_s}$$

a priori in $\mathbb{C}[[\zeta]]$, but the right-hand side is also a series of holomorphic functions and inequalities (8.4) will yield uniform convergence in every compact subset of the principal sheet of $\mathcal{R}(\mathbb{Z})$.

Indeed, let $\rho \in]0, \frac{1}{2}[$. The domain considered in (8.5) consists of those $\zeta \in \mathbb{C}$ such that the segment $[0, \zeta]$ does not meet the open discs $D(-1, \rho)$ and $D(1, \rho)$. All

the $\widehat{\psi}_n$'s are holomorphic in this domain \mathcal{D}_ρ (we had to delete the disc around -1 only because of $\widehat{\psi}_0$).

Since \mathcal{D}_ρ is star-shaped with respect to 0 , the analytic continuation of the convolution product of any two functions $\widehat{\varphi}$ and $\widehat{\psi}$ holomorphic in \mathcal{D}_ρ is defined by formula (8.2) regardless of the size of $|\zeta|$. If moreover one has inequalities of the form $|\widehat{\varphi}(\zeta)| \leq \Phi(|\zeta|) e^{C|\zeta|}$ and $|\widehat{\psi}(\zeta)| \leq \Psi(|\zeta|) e^{C|\zeta|}$ in \mathcal{D}_ρ , then the inequality $|\widehat{\varphi} * \widehat{\psi}(\zeta)| \leq \Phi * \Psi(|\zeta|) e^{C|\zeta|}$ holds in \mathcal{D}_ρ . Hence

$$|\widehat{\varphi}_n(\zeta)| \leq \sum_{\substack{s \geq 1, n_1, \dots, n_s \geq 0 \\ n_1 + \dots + n_s = n+s-1}} \frac{1}{s} \binom{n+s-1}{s-1} K^s L^{n+s-1} M_s(|\zeta|) e^{C|\zeta|}, \quad \zeta \in \mathcal{D}_\rho,$$

with $M_s(\zeta) = 1^{*s}(\zeta) = \frac{\zeta^{s-1}}{(s-1)!}$. The conclusion follows since the right-hand side is less than $K(4L)^n e^{(C+8KL)|\zeta|}$.

9 The $\tilde{\mathcal{V}}^\omega$'s as resurgence monomials—introduction to alien calculus

9.1 Resurgence theory means much more than Borel-Laplace summation. It incorporates a study of the role of the singularities which appear in the Borel plane (i.e. the plane of the complex variable ζ), which can be performed through the so-called *alien calculus*.

We shall now recall Écalle's definitions in a particular case which will suffice for the saddle-node problem. We shall give less details than in the previous section; see e.g. [14], §2.3 for more information (and *op. cit.*, §3 for an outline of the general case and more references).

The reader will thus find in this section the definition of a subalgebra $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ of $\widetilde{\mathbf{R}}_{\mathbb{Z}}$, which is called the algebra of *simple resurgent functions over \mathbb{Z}* , and of a collection of operators Δ_m , $m \in \mathbb{Z}^*$, which are derivations of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ called *alien derivations*. Alien calculus consists in the proper use of these derivations.

We shall see that the formal series $\tilde{\mathcal{V}}^{\omega_1, \dots, \omega_r}$ belong to $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ and study the effect of the alien derivations on them.

9.2 Let $\widehat{\varphi}$ be holomorphic in an open subset U of \mathbb{C} and $\omega \in \partial U$. We say that $\widehat{\varphi}$ has a *simple singularity* at ω if there exist $C \in \mathbb{C}$ and $\widehat{\chi}(\zeta)$, $\text{reg}(\zeta) \in \mathbb{C}\{\zeta\}$ such that

$$\widehat{\varphi}(\zeta) = \frac{C}{2\pi i(\zeta - \omega)} + \frac{1}{2\pi i} \widehat{\chi}(\zeta - \omega) \log(\zeta - \omega) + \text{reg}(\zeta - \omega) \quad (9.1)$$

for ζ close enough to ω . The *residuum* C and the *variation* $\widehat{\chi}$ are then determined by $\widehat{\varphi}$ (independently of the choice of the branch of the logarithm):

$$C = 2\pi i \lim_{\substack{\zeta \rightarrow \omega \\ \zeta \in U}} (\zeta - \omega) \widehat{\varphi}(\zeta), \quad \widehat{\chi}(\zeta) = \widehat{\varphi}(\omega + \zeta) - \widehat{\varphi}(\omega + \zeta e^{-2\pi i}),$$

where it is understood that considering $\omega + \zeta e^{-2\pi i}$ means following the analytic continuation of $\widehat{\varphi}$ along the circular path $t \in [0, 1] \mapsto \omega + \zeta e^{-2\pi i t}$ (which is possible when starting from $\omega + \zeta \in U$ provided $|\zeta|$ is small enough). Let us use the notation

$$\text{sing}_\omega \widehat{\varphi} = C\delta + \widehat{\chi} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}.$$

in this situation.

We recall that $\widehat{\mathbf{R}}_{\mathbb{Z}} = \mathbb{C}\delta \oplus \widehat{H}(\mathcal{R}(\mathbb{Z}))$.

Definition 9.1. A simple resurgent function over \mathbb{Z} is any $c\delta + \widehat{\varphi} \in \widehat{\mathbf{R}}_{\mathbb{Z}}$ such that all branches of the holomorphic function $\widehat{\varphi} \in \widehat{H}(\mathcal{R}(\mathbb{Z}))$ only have simple singularities (necessarily located at points of \mathbb{Z}). The space of simple resurgent functions over \mathbb{Z} will be denoted $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$.

It turns out that $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ is stable by convolution: it is a subalgebra of $\widehat{\mathbf{R}}_{\mathbb{Z}}$. This is the *convolutive model of the algebra of simple resurgent functions*. The formal model is defined as $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}} = \mathcal{B}^{-1}(\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}})$, which is a subalgebra of $\widetilde{\mathbf{R}}_{\mathbb{Z}}$.

9.3 For a simple resurgent function $c\delta + \widehat{\varphi}$ and a path γ which starts from 0 and then avoids \mathbb{Z} , we shall denote by $\text{cont}_\gamma \widehat{\varphi}$ the analytic continuation of $\widehat{\varphi}$ along γ : this function is analytic in a neighbourhood of the endpoint of γ and admits itself an analytic continuation along all the paths which avoid \mathbb{Z} . If the endpoint of γ is close to m (say at a distance $< \frac{1}{2}$), then the singularity $\text{sing}_m(\text{cont}_\gamma \widehat{\varphi}) \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$ is well-defined (notice that it depends on the branch under consideration, i.e. on γ , and not only on m and $\widehat{\varphi}$). It is easy to see that $\text{sing}_m(\text{cont}_\gamma \widehat{\varphi})$ is itself a simple resurgent function; we thus have, for γ and m as above, a \mathbb{C} -linear operator $c\delta + \widehat{\varphi} \mapsto \text{sing}_m(\text{cont}_\gamma \widehat{\varphi})$ from $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ to itself.

Definition 9.2. Let $m \in \mathbb{Z}^*$. If $m \geq 1$, we define an operator from $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ to itself by using 2^{m-1} particular paths γ :

$$\Delta_m(c\delta + \widehat{\varphi}) = \sum_{\varepsilon \in \{+,-\}^{m-1}} \frac{p_\varepsilon! q_\varepsilon!}{m!} \text{sing}_m(\text{cont}_{\gamma_\varepsilon} \widehat{\varphi}) \quad (9.2)$$

where p_ε and $q_\varepsilon = m - 1 - p_\varepsilon$ denote the numbers of signs ‘+’ and of signs ‘-’ in the sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{m-1})$, and the oriented path γ_ε connects 0 and m following the segment $]0, m[$ but circumventing the intermediary integer points k to the right if $\varepsilon_k = +$ and to the left if $\varepsilon_k = -$.

If $m \leq -1$, the \mathbb{C} -linear operator Δ_m is defined similarly, using the $2^{|m|-1}$ paths γ_ε which follow $]0, -|m|[$ but circumvent the intermediary integer points $-k$ to the right if $\varepsilon_k = +$ and to the left if $\varepsilon_k = -$.

Proposition 9.1. For each $m \in \mathbb{Z}^*$, the operator Δ_m is a \mathbb{C} -linear derivation of $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$.

(For the proof, see [3] or [14], §2.3; see also Lemma 9.2 and the comment on it below.)

By conjugacy by the formal Borel transform \mathcal{B} , we get a derivation of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$, still denoted Δ_m since there is no risk of confusion. The operator Δ_m is called the *alien derivation of index m* (either in the convolutive model $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ or in the formal model $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$).

One can easily check from Definition 9.2 that

$$[\partial, \Delta_m] = m\Delta_m \text{ in } \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}, \quad [\widehat{\partial}, \Delta_m] = m\Delta_m \text{ in } \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}, \quad (9.3)$$

where ∂ denotes the natural derivation $\frac{d}{dz}$ of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ and $\widehat{\partial}$ is the corresponding derivation $c\delta + \widehat{\varphi}(\zeta) \mapsto -\zeta\widehat{\varphi}(\zeta)$ of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$.

9.4 We shall see that the operators Δ_m are independent in a strong sense (see Theorem 8 below). This will rely on a study of the way the alien derivations act on the resurgent functions $\widehat{\mathcal{V}}^{\omega_1, \dots, \omega_r}$.

In this article, for the sake of simplicity, we shall not introduce the larger commutative algebras $\widetilde{\text{RES}}^{\text{simp}}$ and $\widetilde{\text{RES}}^{\text{simp}}$ of simple resurgent functions “over \mathbb{C} ”, i.e. with simple singularities in the Borel plane which can be located anywhere. In these algebras act alien derivations indexed by any non-zero complex number. One could easily adapt the arguments that we are about to develop to the study of the alien derivations Δ_ω , $\omega \in \mathbb{C}^*$, in $\widetilde{\text{RES}}^{\text{simp}}$.

One can also define an even larger commutative algebra of resurgent functions, without any restriction on the nature of the singularities to be encountered in the Borel plane, on which act alien derivations Δ_ω indexed by points ω of the Riemann surface of the logarithm, but there is no formal counterpart contained in $\mathbb{C}[[z^{-1}]]$ (see e.g. [14], §3, and the references therein).

9.5 We now check that the formal series $\widetilde{\mathcal{V}}^{\omega_1, \dots, \omega_r}$ are simple resurgent functions and slightly extend at the same time their definition.

Lemma 9.1. *Let $\mathbf{A} = \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ and $\Omega \subset \mathbb{Z}$. Assume that $a = (\widehat{a}_\eta)_{\eta \in \Omega}$ is a family of entire functions; if $0 \in \Omega$, we assume furthermore that $\widehat{a}_0(0) = 0$. Let $\tilde{a}_\eta = \mathcal{B}^{-1}\widehat{a}_\eta \in z^{-1}\mathbb{C}[[z^{-1}]]$.*

Then the equations $\widetilde{\mathcal{V}}_a^\emptyset = \widetilde{\mathcal{V}}_a^\emptyset = 1$ and, for $\omega \in \Omega^\bullet$ non-empty,

$$\left(\frac{d}{dz} + \|\omega\|\right)\widetilde{\mathcal{V}}_a^\omega = \tilde{a}_{\omega_1}\widetilde{\mathcal{V}}_a^{\cdot\omega}, \quad \left(\frac{d}{dz} + \|\omega\|\right)\widetilde{\mathcal{V}}_a^{\omega'} = -\tilde{a}_{\omega_r}\widetilde{\mathcal{V}}_a^{\omega'}, \quad (9.4)$$

(with ‘ ω ’ denoting ω deprived from its first letter, ω' denoting ω deprived from its last letter and $\|\omega\|$ the sum of the letters of ω) determine inductively two moulds $\widetilde{\mathcal{V}}_a^\bullet, \widetilde{\mathcal{V}}_a^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$, which are symmetral and mutually inverse for mould multiplication.

Proof. A mere adaptation of Lemma 3.2 and Proposition 5.5 (in which the fact that $\Omega = \mathcal{N}$ played no role) shows that $\widetilde{\mathcal{V}}_a^\bullet$ and $\widetilde{\mathcal{V}}_a^\bullet$ are well-defined by (9.4) as moulds on Ω with values in $\mathbb{C}[[z^{-1}]]$, with $\widetilde{\mathcal{V}}_a^\omega, \widetilde{\mathcal{V}}_a^{\omega'} \in z^{-1}\mathbb{C}[[z^{-1}]]$ as soon as $\omega \neq \emptyset$, that they are related by the involution S of Proposition 5.2:

$$\widetilde{\mathcal{V}}_a^\bullet = S\widetilde{\mathcal{V}}_a^\bullet \quad (9.5)$$

and symmetral, hence mutually inverse.

The formal Borel transforms are given by $\widehat{\mathcal{V}}_a^\emptyset = \widehat{\mathcal{V}}_a^\omega = \delta$ and, for $\omega \neq \emptyset$,

$$\widehat{\mathcal{V}}_a^\omega(\zeta) = -\frac{1}{\zeta - \|\omega\|} (\widehat{a}_{\omega_1} * \widehat{\mathcal{V}}_a^\omega), \quad \widehat{\mathcal{V}}_a^\omega(\zeta) = \frac{1}{\zeta - \|\omega\|} (\widehat{a}_{\omega_r} * \widehat{\mathcal{V}}_a^\omega), \quad (9.6)$$

where the right-hand sides belong to $\mathbb{C}[[\zeta]]$ even if $\|\omega\| = 0$, by the same argument as in the proof of Lemma 3.2, and in fact to $\mathbb{C}\{\zeta\}$, by induction on $r(\omega)$.

Since $\|\omega\|$ always lies in \mathbb{Z} , we can apply Lemma 8.3: we get $\widehat{\mathcal{V}}_a^\omega, \widehat{\mathcal{V}}_a^\omega \in \widehat{H}(\mathcal{R}(\mathbb{Z}))$ for all $\omega \neq \emptyset$ by induction on $r(\omega)$, hence our moulds take their values in $\widehat{\mathbf{R}}_{\mathbb{Z}}$.

We see that the singularities of $\widehat{\mathcal{V}}_a^\omega$ and $\widehat{\mathcal{V}}_a^\omega$ are all simple singularities, because of the following addendum to Lemma 8.3: with the hypotheses and notations of that lemma, if moreover $\widehat{\varphi} \in \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$, then $\widehat{b} * \widehat{\varphi}$ vanishes at the origin and only has simple singularities with vanishing residuum (this follows from the first formula in (8.14)), hence $\widehat{s}(\widehat{b} * \widehat{\varphi}) \in \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$. \square

Notice that, by iterating (9.6), one gets

$$\widehat{\mathcal{V}}_a^\omega = (-1)^r \frac{1}{\zeta - \widehat{\omega}_1} \left(\widehat{a}_{\omega_1} * \left(\frac{1}{\zeta - \widehat{\omega}_2} \left(\widehat{a}_{\omega_2} * \left(\cdots \left(\frac{1}{\zeta - \widehat{\omega}_r} \widehat{a}_{\omega_r} \right) \cdots \right) \right) \right) \right) \quad (9.7)$$

$$\widehat{\mathcal{V}}_a^\omega = \frac{1}{\zeta - \check{\omega}_r} \left(\widehat{a}_{\omega_r} * \left(\frac{1}{\zeta - \check{\omega}_{r-1}} \left(\widehat{a}_{\omega_{r-1}} * \left(\cdots \left(\frac{1}{\zeta - \check{\omega}_1} \widehat{a}_{\omega_1} \right) \cdots \right) \right) \right) \right) \quad (9.8)$$

with the notation of (8.7). These are iterated integrals; for instance, the second formula can be written

$$\begin{aligned} \widehat{\mathcal{V}}_a^\omega(\zeta) &= \frac{1}{\zeta - \check{\omega}_r} \int \cdots \int_{0 < \zeta_1 < \cdots < \zeta_{r-1} < \zeta} \widehat{a}_{\omega_r}(\zeta - \zeta_{r-1}) \cdot \\ &\quad \cdot \frac{\widehat{a}_{\omega_{r-1}}(\zeta_{r-1} - \zeta_{r-2})}{\zeta_{r-1} - \check{\omega}_{r-1}} \cdots \frac{\widehat{a}_{\omega_2}(\zeta_2 - \zeta_1)}{\zeta_2 - \check{\omega}_2} \frac{\widehat{a}_{\omega_1}(\zeta_1)}{\zeta_1 - \check{\omega}_1} d\zeta_1 \cdots d\zeta_{r-1} \end{aligned} \quad (9.9)$$

and its analytic continuation along any parametrised path γ which starts from 0 and then avoids \mathbb{Z} is given by the same integral, but taken over all $(r-1)$ -tuples $(\zeta_1, \dots, \zeta_{r-1}) = (\gamma_\varepsilon(t_1), \dots, \gamma_\varepsilon(t_{r-1}))$ with $t_1 < \cdots < t_{r-1}$.

9.6 We are now ready to study the alien derivatives of the resurgent functions $\widehat{\mathcal{V}}_a^{\omega_1, \dots, \omega_r}$ and $\widetilde{\mathcal{V}}_a^{\omega_1, \dots, \omega_r}$ (in the formal model as well as in the convolutive model, the difference is immaterial here).

Proposition 9.2. *Let $\Omega \subset \mathbb{Z}$ and $a = (\widehat{a}_\eta)_{\eta \in \Omega}$ as in Lemma 9.1. For each $m \in \mathbb{Z}^*$, denote by the same symbol Δ_m the alien derivation of index m on $\mathbf{A} = \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ and the mould derivation it induces on $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ by (4.4). Then there exists a scalar-valued altermal mould $V_a^\bullet(m) \in \mathcal{M}^\bullet(\Omega, \mathbb{C})$ such that*

$$\Delta_m \widetilde{\mathcal{V}}_a^\bullet = \widetilde{\mathcal{V}}_a^\bullet \times V_a^\bullet(m), \quad \Delta_m \widetilde{\mathcal{V}}_a^\bullet = -V_a^\bullet(m) \times \widetilde{\mathcal{V}}_a^\bullet. \quad (9.10)$$

Moreover, if $\omega \in \Omega^\bullet$ is non-empty,

$$\|\omega\| \neq m \Rightarrow V_a^\omega(m) = 0. \quad (9.11)$$

Proof. Since $\tilde{\mathcal{V}}_a^\bullet$ and $S\tilde{\mathcal{V}}_a^\bullet = \tilde{\mathcal{V}}_a^\bullet$ are mutually inverse, we get

$$\Delta_m \tilde{\mathcal{V}}_a^\bullet = \tilde{\mathcal{V}}_a^\bullet \times \tilde{V}_a^\bullet(m), \quad \Delta_m \tilde{\mathcal{V}}_a^\bullet = \tilde{V}_a^\bullet(m) \times \tilde{\mathcal{V}}_a^\bullet$$

by defining the moulds $\tilde{V}_a^\bullet(m)$ and $\tilde{V}_a^\bullet(m)$ as

$$\tilde{V}_a^\bullet(m) = \tilde{\mathcal{V}}_a^\bullet \times \Delta_m \tilde{\mathcal{V}}_a^\bullet, \quad \tilde{V}_a^\bullet(m) = \Delta_m \tilde{\mathcal{V}}_a^\bullet \times \tilde{\mathcal{V}}_a^\bullet,$$

but a priori all these moulds take their values in \mathbf{A} . The operators Δ_m and S clearly commute, thus $\tilde{V}_a^\bullet(m) = S\tilde{V}_a^\bullet(m)$. Proposition 5.4 shows that $\tilde{V}_a^\bullet(m)$ and $\tilde{V}_a^\bullet(m)$ are altermal; Proposition 5.2 then shows that they are opposite of one another: $\tilde{V}_a^\bullet(m) = -\tilde{V}_a^\bullet(m)$.

It only remains to be checked that $\tilde{V}_a^\bullet(m)$ is scalar-valued and satisfies (9.11). This will follow from the equation

$$(\partial + \nabla - m)\tilde{V}_a^\bullet(m) = 0, \quad (9.12)$$

where ∂ denotes the differential $\frac{d}{dz}$ as well as the mould derivation it induces by (4.4) and ∇ is the mould derivation (4.3).

Here is the proof of (9.12): $\tilde{\mathcal{V}}_a^\bullet$ is defined on non-empty words by the first equation in (9.4), which can be written

$$(\partial + \nabla)\tilde{\mathcal{V}}_a^\bullet = \tilde{J}_a^\bullet \times \tilde{\mathcal{V}}_a^\bullet, \quad (9.13)$$

with $\tilde{J}_a^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$ defined exactly as in (4.6). Let us apply the derivation Δ_m to both sides of equation (9.13), using $\Delta_m(\partial + \nabla) = (\partial + \nabla - m)\Delta_m$ (consequence of (9.3) and of $[\nabla, \Delta_m] = 0$) and $\Delta_m \tilde{J}_a^\bullet = 0$ (consequence of the vanishing of Δ_m on entire functions):

$$(\partial + \nabla - m)\Delta_m \tilde{\mathcal{V}}_a^\bullet = \tilde{J}_a^\bullet \times \Delta_m \tilde{\mathcal{V}}_a^\bullet.$$

Writing $\Delta_m \tilde{\mathcal{V}}_a^\bullet$ as $\tilde{\mathcal{V}}_a^\bullet \times \tilde{V}_a^\bullet(m)$ and using the fact that $\partial + \nabla$ is a derivation, we get

$$((\partial + \nabla)\tilde{\mathcal{V}}_a^\bullet) \times \tilde{V}_a^\bullet(m) + \tilde{\mathcal{V}}_a^\bullet \times (\partial + \nabla - m)\tilde{V}_a^\bullet(m) = \tilde{J}_a^\bullet \times \tilde{\mathcal{V}}_a^\bullet \times \tilde{V}_a^\bullet(m),$$

whence $\tilde{\mathcal{V}}_a^\bullet \times (\partial + \nabla - m)\tilde{V}_a^\bullet(m) = 0$ by a further use of (9.13). Since $\tilde{\mathcal{V}}_a^\emptyset = 1$ and $\mathcal{M}^\bullet(\Omega, \mathbf{A})$ is an integral domain, this yields (9.12).

We conclude the proof by interpreting this relation in the convolutive model: we already knew that $\tilde{V}_a^\emptyset(m) = 0$; now, for any non-empty ω , we have

$$\mathcal{B}\tilde{V}_a^\omega(m) = V_a^\omega(m)\delta + \hat{V}_a^\omega(m)(\zeta)$$

with $V_a^\omega(m) \in \mathbb{C}$ and $\hat{V}_a^\omega(m) \in \hat{H}(\mathcal{R}(\mathbb{Z}))$ satisfying

$$(\|\omega\| - m)V_a^\omega(m) = 0, \quad (-\zeta + \|\omega\| - m)\hat{V}_a^\omega(m) = 0,$$

whence $V_a^\omega(m) = 0$ for $\|\omega\| \neq m$ and $\hat{V}_a^\omega(m) = 0$ for all ω (since both \mathbb{C} and $\hat{H}(\mathcal{R}(\mathbb{Z})) \subset \mathbb{C}\{\zeta\}$ are integral domains). \square

9.7 Formulas (9.10), when evaluated in the convolutive model on $\omega \in \Omega^\bullet$, read

$$\Delta_m \widehat{\mathcal{V}}_a^\emptyset = \Delta_m \widehat{\mathcal{V}}_a^\emptyset = 0$$

for $r(\omega) = 0$, which is obvious since $\widehat{\mathcal{V}}_a^\emptyset = \widehat{\mathcal{V}}_a^\emptyset = \delta$. For $r(\omega) = 1$, we get

$$\Delta_m \widehat{\mathcal{V}}_a^{\omega_1} = -\Delta_m \widehat{\mathcal{V}}_a^{\omega_1} = V_a^{\omega_1}(m) \delta$$

and the explicit value of the coefficient is

$$\omega_1 = m \Rightarrow V_a^{\omega_1}(m) = -2\pi i \widehat{a}_m(m), \quad \omega_1 \neq m \Rightarrow V_a^{\omega_1}(m) = 0. \quad (9.14)$$

This is a simple residuum computation for the meromorphic functions $\widehat{\mathcal{V}}_a^{\omega_1}(\zeta) = -\widehat{\mathcal{V}}_a^{\omega_1}(\zeta) = -\frac{\widehat{a}_{\omega_1}(\zeta)}{\zeta - \omega_1}$ (observe that the value of \widehat{a}_m at m and thus of $V_a^m(m)$ depend transcendentally on the Taylor coefficients of \widehat{a}_m at the origin).

For $r = r(\omega) \geq 2$, we get

$$\begin{aligned} \Delta_m \widehat{\mathcal{V}}_a^\omega &= V_a^\omega(m) \delta + \sum_{i=1}^{r-1} V_a^{\omega_{i+1}, \dots, \omega_r}(m) \widehat{\mathcal{V}}_a^{\omega_1, \dots, \omega_i}, \\ \Delta_m \widehat{\mathcal{V}}_a^\omega &= -V_a^\omega(m) \delta - \sum_{i=1}^{r-1} V_a^{\omega_1, \dots, \omega_i}(m) \widehat{\mathcal{V}}_a^{\omega_{i+1}, \dots, \omega_r}. \end{aligned} \quad (9.15)$$

The number $V_a^\omega(m)$ thus appears as the *residuum* of a certain simple singularity, which is a combination of the singularities at m of certain branches of $\widehat{\mathcal{V}}_a^\omega$ or $\widehat{\mathcal{V}}_a^\omega$; on the other hand, the fact that the *variation* of this singularity can be expressed as a linear combination of the functions $\widehat{\mathcal{V}}_a^{\omega_1, \dots, \omega_i}$ or $\widehat{\mathcal{V}}_a^{\omega_{i+1}, \dots, \omega_r}$ is related to the very origin of the name “resurgent functions”: the functions $\widehat{\mathcal{V}}_a^\omega(\zeta)$ or $\widehat{\mathcal{V}}_a^\omega(\zeta)$, which were initially defined for ζ close to the origin by (9.7)–(9.8), “resurrect” in the variation of the singularities of their analytic continuations.

An even more striking instance of this “resurgence phenomenon” is the Bridge Equation, to be discussed in the case of the saddle-node problem in Section 10 below.

9.8 The computation of the number $V_a^\omega(m)$ is not as easy when $r(\omega) \geq 2$ as in the case $r = 1$.

First observe that the vanishing of $V_a^\omega(m)$ when $\|\omega\| = \hat{\omega}_1 = \check{\omega}_r \neq m$ could be obtained as a consequence of the analytic continuation of formulas (9.7)–(9.8) (for instance, the singularities of the analytic continuation of $\widehat{\mathcal{V}}_a^\omega$ can only be located at $\check{\omega}_1, \dots, \check{\omega}_r$ and, among them, only the one at $\check{\omega}_r$ can have a non-zero residuum—cf. the argument at the end of the proof of Lemma 9.1).

For $\|\omega\| = m$, using the notations of Definition 9.2, one can write $V_a^\omega(m)$ as a combination of iterated integrals: (9.9) and (9.15) yield

$$\begin{aligned} V_a^\omega(m) = -2\pi i \sum_{\varepsilon \in \{+,-\}^{|m|-1}} \frac{p_\varepsilon! q_\varepsilon!}{|m|!} \int_{\Gamma_\varepsilon} \widehat{a}_{\omega_r}(m - \zeta_{r-1}) \cdot \\ \cdot \frac{\widehat{a}_{\omega_{r-1}}(\zeta_{r-1} - \zeta_{r-2})}{\zeta_{r-1} - \check{\omega}_{r-1}} \cdots \frac{\widehat{a}_{\omega_2}(\zeta_2 - \zeta_1)}{\zeta_2 - \check{\omega}_2} \frac{\widehat{a}_{\omega_1}(\zeta_1)}{\zeta_1 - \check{\omega}_1} d\zeta_1 \cdots d\zeta_{r-1}, \quad (9.16) \end{aligned}$$

where Γ_ε consists of all $(r-1)$ -tuples $(\zeta_1, \dots, \zeta_{r-1}) = (\gamma_\varepsilon(t_1), \dots, \gamma_\varepsilon(t_{r-1}))$ with $t_1 < \dots < t_{r-1}$, for any parametrisation of the oriented path γ_ε (which connects 0 and $m = \check{\omega}_r$). In fact, one can restrict oneself to the paths which follow the segment $[0, m[$ circumventing the points of $\{\check{\omega}_1, \dots, \check{\omega}_{r-1}\} \cap]0, m[= \{k_1, \dots, k_s\}$ to the right or to the left, labelled by sequences $\varepsilon \in \{+,-\}^s$, with weights $p_\varepsilon! q_\varepsilon! / (s+1)!$.

The formula gets simpler when $\Omega \subset \mathbb{Z}^*$ and $\tilde{a}_\eta \equiv z^{-1}$ for each $\eta \in \Omega$, since each \widehat{a}_η is then the constant function with value 1:

$$\|\omega\| = m \Rightarrow V_a^\omega(m) = -2\pi i \sum_{\varepsilon \in \{+,-\}^{|m|-1}} \frac{p_\varepsilon! q_\varepsilon!}{|m|!} \int_{\Gamma_\varepsilon} \frac{d\zeta_1 \cdots d\zeta_{r-1}}{(\zeta_1 - \check{\omega}_1) \cdots (\zeta_{r-1} - \check{\omega}_{r-1})}.$$

In this last case, the numbers $V_a^\omega(m)$ are connected with multiple logarithms. They are studied under the name “canonical hyperlogarithmic mould” in [3], chap. 7, without the restriction $\Omega \subset \mathbb{Z}$ (which we imposed here only to avoid having to define the larger algebra $\widehat{\text{RES}}^{\text{simp}}$; also the condition $0 \notin \Omega$ was imposed here only to simplify the discussion).

Observe that $V_a^\bullet(m)$ is always a primitive element of the graded cocommutative Hopf algebra $\mathcal{H}^\bullet(\Omega, \mathbb{C})$ defined in Section 5 (this is just a rephrasing of the shuffle relations encoded by the alternality of this scalar mould).

9.9 Formulas (9.10) can be iterated so as to express all the successive alien derivatives of our resurgent functions $\tilde{\mathcal{V}}_a^\omega$ or $\tilde{\mathcal{V}}_a^\bullet$:

$$\begin{aligned} \Delta_{m_s} \cdots \Delta_{m_1} \tilde{\mathcal{V}}_a^\bullet &= \tilde{\mathcal{V}}_a^\bullet \times V_a^\bullet(m_s) \times \cdots \times V_a^\bullet(m_1), \\ \Delta_{m_s} \cdots \Delta_{m_1} \tilde{\mathcal{V}}_a^\bullet &= (-1)^s V_a^\bullet(m_1) \times \cdots \times V_a^\bullet(m_s) \times \tilde{\mathcal{V}}_a^\bullet, \quad (9.17) \end{aligned}$$

for $s \geq 1$ and $m_1, \dots, m_s \in \mathbb{Z}^*$.

We can consider the collection of resurgent functions $(\tilde{\mathcal{V}}_a^\omega)_{\omega \in \Omega^\bullet}$ (or $(\tilde{\mathcal{V}}_a^\bullet)_{\omega \in \Omega^\bullet}$) as closed under alien derivation (i.e. all their alien derivatives can be expressed through relations involving themselves and scalars); it was already closed under multiplication (by symmetrality), and even under ordinary differentiation, in view of (9.4), if we admit relations with coefficients in $\mathbb{C}\{z^{-1}\}$ (but, after all, convergent series can be considered as “resurgent constants”: all alien derivations act trivially on them).

This is why the $\tilde{\mathcal{V}}_a^\omega$'s are called “resurgent monomials”: they behave nicely under elementary operations such as multiplication and alien derivations. In fact, in Section 12 below, we shall deduce from them another family of resurgence

monomials which behave even better under the action of alien derivations (but the price to pay is that their ordinary derivatives are not as simple as (9.4)).

Notice that the operator $\Delta_{m_s} \cdots \Delta_{m_1}$ measures a combination of singularities located at $m_1 + \cdots + m_s$. For instance, the fact that $V_a^\bullet(m_s) \times \cdots \times V_a^\bullet(m_1)$ vanishes on any word ω such that $\|\omega\| \neq m_1 + \cdots + m_s$ (easy consequence of (9.11)) is consistent with the vanishing of the residuum at any point $\neq \hat{\omega}_1$ of any branch of $\hat{\mathcal{V}}_a^\omega$ (consequence of the analytic continuation of (9.7)).

9.10 Let $\Omega \subset \mathbb{Z}$ and $a = (\hat{a}_\eta)_{\eta \in \Omega}$ be a family of entire functions as in Lemma 9.1, thus with $\hat{a}_0(0) = 0$ if $0 \in \Omega$. We end this section by illustrating mould calculus to derive quadratic shuffle relations for the numbers

$$L_a^\omega = 2\pi i \int_{\Gamma^+} \hat{a}_{\omega_r}(\|\omega\| - \zeta_{r-1}) \frac{\hat{a}_{\omega_{r-1}}(\zeta_{r-1} - \zeta_{r-2})}{\zeta_{r-1} - \check{\omega}_{r-1}} \cdots \frac{\hat{a}_{\omega_2}(\zeta_2 - \zeta_1)}{\zeta_2 - \check{\omega}_2} \frac{\hat{a}_{\omega_1}(\zeta_1)}{\zeta_1 - \check{\omega}_1} d\zeta_1 \cdots d\zeta_{r-1}, \quad (9.18)$$

for $\omega \in \Omega^\bullet$ non-empty, where $\Gamma^+ = \Gamma_\varepsilon$ with $\varepsilon = (+, \dots, +) \in \{+, -\}^{|m|-1}$ for $m = \|\omega\|$ (notation of (9.16); if $r = 1$, then $L_a^{\omega_1} = 2\pi i \hat{a}_{\omega_1}(\omega_1)$). This includes the case of the multiple logarithms

$$L^\omega = 2\pi i \int_{\Gamma^+} \frac{d\zeta_1 \cdots d\zeta_{r-1}}{(\zeta_1 - \check{\omega}_1) \cdots (\zeta_{r-1} - \check{\omega}_{r-1})},$$

with $\omega_1, \dots, \omega_r \in \Omega \subset \mathbb{Z}^*$ (obtained when $\hat{a}_\eta(\zeta) \equiv 1$).¹²

It is convenient to use here the auxiliary operators Δ_m^+ of $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ defined by the formulas $\Delta_0^+ = \text{Id}$ and, for $m \in \mathbb{Z}^*$,

$$\Delta_m^+(c\delta + \hat{\varphi}) = \text{sing}_m(\text{cont}_{\gamma^+} \hat{\varphi}), \quad (9.19)$$

where $\gamma^+ = \gamma_\varepsilon$ with $\varepsilon = (+, \dots, +) \in \{+, -\}^{|m|-1}$. Thus

$$L_a^\omega = \text{coefficient of } \delta \text{ in } \Delta_{\|\omega\|}^+ \hat{\mathcal{V}}_a^\omega. \quad (9.20)$$

We shall consider L_a^ω as the value at ω of a scalar mould L_a^\bullet ; we set $L_a^\emptyset = 1$, so that (9.20) still holds when $\omega = \emptyset$.

Proposition 9.3. *The numbers L_a^ω satisfy the shuffle relations*

$$\sum_{\omega \in \Omega^\bullet} \text{sh} \begin{pmatrix} \omega^1, \omega^2 \\ \omega \end{pmatrix} L_a^\omega = \begin{cases} L_a^{\omega^1} L_a^{\omega^2} & \text{if } \|\omega^1\| \cdot \|\omega^2\| \geq 0 \\ 0 & \text{if not} \end{cases}$$

for any non-empty $\omega^1, \omega^2 \in \Omega^\bullet$. Equivalently, the scalar moulds $L_{a,\pm}^\bullet$ defined by

$$L_{a,+}^\omega = 1_{\{\|\omega\| \geq 0\}} L_a^\omega, \quad L_{a,-}^\omega = 1_{\{\|\omega\| \leq 0\}} L_a^\omega \quad (9.21)$$

(for any $\omega \in \Omega^\bullet$, with the convention $\|\emptyset\| = 0$) are symmetrical.

This can be rephrased by saying that $L_{a,+}^\bullet$ and $L_{a,-}^\bullet$ are group-like elements of the graded cocommutative Hopf algebra $\mathcal{H}^\bullet(\Omega, \mathbb{C})$ defined in Section 5.

¹²We recall that $\check{\omega}_1 = \omega_1, \check{\omega}_2 = \omega_1 + \omega_2, \dots, \check{\omega}_{r-1} = \omega_1 + \cdots + \omega_{r-1}$ (thus L^ω depends on ω_r only through Γ^+ which connects the origin and $\check{\omega}_r$).

The rest of this section is devoted to the proof of Proposition 9.3. We begin by a few facts about the operators Δ_m^+ ; these are not derivations, as the alien derivations Δ_m , but they are related to them and satisfy modified Leibniz rules analogous to (7.4):

Lemma 9.2. *The operators Δ_m^+ defined in (9.19) are related to the alien derivations (9.2) by the following relations: for any $m \in \mathbb{Z}^*$,*

$$\Delta_m^+ = \sum \frac{1}{s!} \Delta_{m_s} \cdots \Delta_{m_1}, \quad \Delta_m = \sum \frac{(-1)^{s-1}}{s} \Delta_{m_s}^+ \cdots \Delta_{m_1}^+, \quad (9.22)$$

with both sums taken over all $s \geq 1$ and $m_1, \dots, m_s \in \mathbb{Z}^*$ of the same sign as m such that $m_1 + \cdots + m_s = m$ (these are thus finite sums). Moreover, for any $\widehat{\chi}_1, \widehat{\chi}_2 \in \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ and $m \in \mathbb{Z}$,

$$\Delta_m^+(\widehat{\chi}_1 * \widehat{\chi}_2) = \sum \Delta_{m_1}^+ \widehat{\chi}_1 * \Delta_{m_2}^+ \widehat{\chi}_2 \quad (9.23)$$

with summation over all $m_1, m_2 \in \mathbb{Z}$ of the same sign as m (but possibly vanishing) such that $m_1 + m_2 = m$.

Let us denote by the same symbols the operators of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ obtained from the Δ_m^+ 's by conjugacy by the formal Borel transform \mathcal{B} , as we did for the Δ_m 's. If we consider the algebras $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}[[e^{-z}]]$ and $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}[[e^z]]$, formula (9.22) can be written

$$\sum_{m \geq 0} e^{-mz} \Delta_m^+ = \exp \left(\sum_{m > 0} e^{-mz} \Delta_m \right), \quad \sum_{m \leq 0} e^{-mz} \Delta_m^+ = \exp \left(\sum_{m < 0} e^{-mz} \Delta_m \right). \quad (9.24)$$

We do not give the proof of this lemma here; see e.g. [14], Lemmas 4 and 5 (the coefficients $p_\varepsilon! q_\varepsilon! / |m|!$ in Definition 9.2 were chosen exactly so that (9.22) hold; the standard properties of the logarithm and exponential series then show that (9.23) and Proposition 9.1 are equivalent; it is in fact easy to check first (9.23) by deforming the contour of integration in the integral giving $\widehat{\chi}_1 * \widehat{\chi}_2$, and then to deduce Proposition 9.1).

Lemma 9.3. *For any $m \in \mathbb{Z}^*$, define a scalar mould $L_a^\bullet(m)$ by the formula*

$$L_a^\bullet(m) = \sum \frac{(-1)^s}{s!} V_a^\bullet(m_1) \times \cdots \times V_a^\bullet(m_s),$$

with summation over all $s \geq 1$ and $m_1, \dots, m_s \in \mathbb{Z}^*$ of the same sign as m such that $m_1 + \cdots + m_s = m$. Define also $L_a^\bullet(0) = 1^\bullet$. Then, for every $m \in \mathbb{Z}$,

- (i) $\Delta_m^+ \widehat{\mathcal{V}}_a^\bullet = L_a^\bullet(m) \times \widehat{\mathcal{V}}_a^\bullet$,
- (ii) $\tau(L_a^\bullet(m)) = \sum L_a^\bullet(m_1) \otimes L_a^\bullet(m_2)$, with summation over all $m_1, m_2 \in \mathbb{Z}$ of the same sign as m such that $m_1 + m_2 = m$,
- (iii) $m = \|\omega\| \Rightarrow L_a^\omega(m) = L_a^\omega, \quad m \neq \|\omega\| \Rightarrow L_a^\omega(m) = 0$ (for any $\omega \in \Omega^\bullet$, with the convention $\|\emptyset\| = 0$).

Proof. The first property follows from (9.17) and (9.22). For the second, we write the symmetrality of $\widehat{\mathcal{V}}_a^\bullet$ and $\widehat{\mathcal{V}}_a^\bullet$ as identities in $\widehat{\mathcal{M}}^{\bullet\bullet}(\Omega, \text{RES}_{\mathbb{Z}}^{\text{simp}})$:

$$\tau(\widehat{\mathcal{V}}_a^\bullet) = \widehat{\mathcal{V}}_a^\bullet \otimes \widehat{\mathcal{V}}_a^\bullet, \quad \tau(\widehat{\mathcal{V}}_a^\bullet) = \widehat{\mathcal{V}}_a^\bullet \otimes \widehat{\mathcal{V}}_a^\bullet,$$

the operator Δ_m^+ induces operators acting on moulds and dimoulds which clearly satisfy $\Delta_m^+ \circ \tau(\widehat{\mathcal{V}}_a^\bullet) = \tau(\Delta_m^+ \widehat{\mathcal{V}}_a^\bullet)$ and relation (9.23) implies

$$\tau(\Delta_m^+ \widehat{\mathcal{V}}_a^\bullet) = \sum_{\substack{m=m_1+m_2 \\ m_i m \geq 0}} \Delta_{m_1}^+ \widehat{\mathcal{V}}_a^\bullet \otimes \Delta_{m_2}^+ \widehat{\mathcal{V}}_a^\bullet,$$

whence the result follows since $\tau(L_a^\bullet(m)) = \tau(\Delta_m^+ \widehat{\mathcal{V}}_a^\bullet) \times \tau(\widehat{\mathcal{V}}_a^\bullet)$ by the homomorphism property of τ applied to (i).

The second part of the third property is obvious when $m = 0$ and follows from (9.11) when $m \neq 0$, because $\|\omega\| \neq m_1 + \dots + m_s$ implies that $V_a^\bullet(m_1) \times \dots \times V_a^\bullet(m_s)$ vanishes on ω (even if $\omega = \emptyset$). The first part of the third property follows from (9.20), since property (i) yields

$$\Delta_m^+ \widehat{\mathcal{V}}_a^\omega = L_a^\omega(m) \delta + \sum_{i=1}^{r-1} L_a^{\omega_1, \dots, \omega_i}(m) \widehat{\mathcal{V}}_a^{\omega_{i+1}, \dots, \omega_r}$$

if $r = r(\omega) \geq 1$ and $\Delta_m^+ \widehat{\mathcal{V}}_a^\emptyset = L_a^\emptyset(m) \delta$ if $\omega = \emptyset$. \square

Proof of Proposition 9.3. We have $L_a^\emptyset = 1$. Let $\omega^1, \omega^2 \in \Omega^\bullet$ be non-empty. Property (ii) with $m = \|\omega^1\| + \|\omega^2\|$ yields

$$\sum_{\omega \in \Omega^\bullet} \text{sh} \begin{pmatrix} \omega^1, \omega^2 \\ \omega \end{pmatrix} L_a^\omega(m) = \sum_{\substack{m=m_1+m_2 \\ m_i m \geq 0}} L_a^{\omega^1}(m_1) L_a^{\omega^2}(m_2).$$

According to Property (iii), the left-hand side is $\tau(L_a^\bullet)^{\omega^1, \omega^2}$ (because any nonzero term in it has $\|\omega\| = m$). Among the $|m| + 1$ terms of the right-hand side, at most one may be nonzero: if $\|\omega^1\|$ and $\|\omega^2\|$ have the same sign, then the term corresponding to $m_1 = \|\omega^1\|$ is $L_a^{\omega^1} L_a^{\omega^2}$ while all the others vanish; but in the opposite case, this term does not belong to the summation and one gets 0 as right-hand side. This is the desired shuffle relation; we leave it to the reader to interpret it in terms of symmetrality for the moulds $L_{a,\pm}^\bullet$ by distinguishing the four possible cases: $\|\omega^1\| \cdot \|\omega^2\| \geq 0$ or < 0 , and $\|\omega^1\| + \|\omega^2\| \geq 0$ or < 0 . \square

In fact, we can write

$$L_{a,+}^\bullet = \sum_{m \geq 0} L_a^\bullet(m) = \exp(-V_{a,+}^\bullet), \quad V_{a,+}^\bullet = \sum_{m > 0} V_a^\bullet(m) \quad (9.25)$$

$$L_{a,-}^\bullet = \sum_{m \leq 0} L_a^\bullet(m) = \exp(-V_{a,-}^\bullet), \quad V_{a,-}^\bullet = \sum_{m < 0} V_a^\bullet(m), \quad (9.26)$$

with well-defined alternal moulds $V_{a,\pm}^\bullet$ (and using \exp as a short-hand for E_1 —see (4.9)), since Lemma 9.3 (iii) and property (9.11) imply that, when evaluated

on a given word ω , these formulas involve only finitely many terms; one could thus have invoked Proposition 5.1 to deduce the symmetrality of $L_{a,\pm}^\bullet$.

10 The Bridge Equation for the saddle-node

In this section, returning to the saddle-node problem, we shall explain why the formal series $\tilde{\varphi}_n(z) = \varphi_n(-1/z)$ and $\tilde{\psi}_n(z) = \psi_n(-1/z)$ of Theorem 2, which were proved to belong to $\tilde{\mathbf{R}}_\mathbb{Z}$, are in fact simple resurgent functions. Moreover, we shall express their alien derivatives in terms of themselves and of the numbers $V_a^\bullet(m)$ of Proposition 9.2.

10.1 We recall the hypotheses and the notations for the saddle-node:

$$X = x^2 \frac{\partial}{\partial x} + A(x, y) \frac{\partial}{\partial y}$$

with $A(x, y) = y + \sum_{\eta \in \Omega} a_\eta(x) y^{\eta+1} \in \mathbb{C}\{x, y\}$, where $\Omega = \{\eta \in \mathbb{Z} \mid \eta \geq -1\}$, $\tilde{a}_\eta(z) = a_\eta(-1/z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $\tilde{a}_0(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$.

We also recall that $\eta \in \Omega \mapsto B_\eta = y^{\eta+1} \frac{\partial}{\partial y}$ gives rise to a comould \mathbf{B}_\bullet such that $\mathbf{B}_\omega y = \beta_\omega y^{\|\omega\|+1}$, where the numbers β_ω , $\omega \in \Omega^\bullet$, satisfy Lemma 8.1 (we define $\beta_\emptyset = 1$ and $\|\emptyset\| = 0$). We set $a = (\mathcal{B}\tilde{a}_\eta)_{\eta \in \Omega}$, so as to be able to make use of the constants $V_a^\omega(m)$, $(m, \omega) \in \mathbb{Z}^* \times \Omega^\bullet$ defined in Proposition 9.2 and more explicitly by formulas (9.14) and (9.16). Later in this section we shall prove

Proposition 10.1. *The family of complex numbers $(\beta_\omega V_a^\omega(m))_{\omega \in \Omega^\bullet, \|\omega\|=m}$ is summable for each $m \in \mathbb{Z}^*$. Let*

$$C_m = \sum_{\omega \in \Omega^\bullet, \|\omega\|=m} \beta_\omega V_a^\omega(m), \quad m \in \mathbb{Z}^*. \quad (10.1)$$

Then $C_m = 0$ for $m \leq -2$.

We call *Écalle's invariants* of X the complex numbers $C_{-1}, C_1, C_2, \dots, C_m, \dots$ because of their role in the Bridge Equation (Theorem 3 below) and in the classification problem (Theorem 5 and Section 11 below).

The formal transformations $\theta(x, y) = (x, \varphi(x, y))$ and $\theta^{-1}(x, y) = (x, \psi(x, y))$ which conjugate X to its normal form $X_0 = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ were constructed in the first part of this article through mould-comould expansions for the corresponding substitution operators Θ and Θ^{-1} . Passing to the resurgence variable $z = -1/x$, we set

$$\tilde{\varphi}(z, y) = \varphi(-1/z, y) = y + \sum_{n \geq 0} \tilde{\varphi}_n(z) y^n, \quad \tilde{\psi}(z, y) = \psi(-1/z, y) = y + \sum_{n \geq 0} \tilde{\psi}_n(z) y^n,$$

where the coefficients $\tilde{\varphi}_n(z)$ and $\tilde{\psi}_n(z)$ are known to belong to the algebra $\tilde{\mathbf{R}}_\mathbb{Z}$ of resurgent functions, by Theorem 2. We also introduce the substitution operator

$$\tilde{\Theta}: \tilde{f}(z, y) \mapsto \tilde{f}(z, \tilde{\varphi}(z, y)) \quad (10.2)$$

(a priori defined in $\mathbb{C}[[z^{-1}, y]]$). Later in this section, we shall prove

Theorem 3. *The formal series $\tilde{\varphi}_n(z)$ and $\tilde{\psi}_n(z)$ are simple resurgent functions, thus $\tilde{\varphi}(z, y)$ and $\tilde{\psi}(z, y)$ belong in fact to $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}[[y]]$.*

Moreover, for any $m \in \mathbb{Z}^$, the formal series of $\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}[[y]]$*

$$\Delta_m \tilde{\varphi} := \sum_{n \geq 0} (\Delta_m \tilde{\varphi}_n) y^n, \quad \Delta_m \tilde{\psi} := \sum_{n \geq 0} (\Delta_m \tilde{\psi}_n) y^n$$

are given by the formulas

$$\Delta_m \tilde{\varphi} = C_m y^{m+1} \frac{\partial \tilde{\varphi}}{\partial y}, \quad \Delta_m \tilde{\psi} = -C_m \tilde{\psi}^{m+1}, \quad m \in \mathbb{Z}^*. \quad (10.3)$$

10.2 The two equations in (10.3) are equivalent forms of the so-called *Bridge Equation*, here expressed in $\mathbf{A}[[y]]$ with $\mathbf{A} = \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$. On the one hand, the left-hand sides represent the action of the alien derivation Δ_m of $\mathbf{A}[[y]]$ (we denote by the same symbol the alien derivation Δ_m of \mathbf{A} and the operator it induces in $\mathbf{A}[[y]]$ by acting separately on each coefficient). On the other hand, both right-hand sides can be expressed with the help of the ordinary differential operator

$$\mathcal{C}(m) = C_m y^{m+1} \frac{\partial}{\partial y},$$

yielding

$$\Delta_m \tilde{\varphi} = \mathcal{C}(m) \tilde{\varphi} = \mathcal{C}(m) \tilde{\Theta} y, \quad (10.4)$$

$$\Delta_m \tilde{\psi} = -\tilde{\Theta}^{-1} \mathcal{C}(m) y. \quad (10.5)$$

See the end of this section for more symmetric formulations of the Bridge Equation, which involve only the operators $\tilde{\Theta}$ or $\tilde{\Theta}^{-1}$ and Δ_m for the left-hand sides, and $\mathcal{C}(m)$ for the right-hand sides.

The name “Bridge Equation” refers to the link thus established between alien and ordinary differential calculus when dealing with the solutions $\tilde{\varphi}$ and $\tilde{\psi}$ of our formal normalisation problem (or with the operator Θ solution of the conjugacy equation (3.1)).

This is a very general phenomenon, in which one sees the advantage of measuring the singularities in the Borel plane through *derivations*: we are dealing with the solutions of non-linear equations (e.g. $(\partial + y \frac{\partial}{\partial y}) \tilde{\varphi}(z, y) = A(-1/z, \tilde{\varphi}(z, y))$ in $\mathbb{C}[[z^{-1}, y]]$), and their alien derivatives must satisfy equations corresponding to the linearisation of these equations; it is thus natural that these alien derivatives can be expressed in terms of the ordinary derivatives of the solutions.

The above argument could be used to derive the form of equation (10.3)¹³, however, in the proof below, we prefer to use the explicit mould representations involving $\tilde{\mathcal{V}}^\bullet$ and $\tilde{\mathcal{V}}^\bullet$ so as to obtain formulas (10.1) for the coefficients C_m .

¹³Compare the linear equations $L \partial_y \tilde{\varphi} = 0$ and $(L - m - 1) \Delta_m \tilde{\varphi} = 0$ where

$$L = \tilde{X}_0 + \tilde{\lambda}(z, y), \quad \tilde{\lambda}(z, y) = 1 - \partial_y A(-1/z, \tilde{\varphi}(z, y)), \quad \tilde{X}_0 = \partial + y \frac{\partial}{\partial y}.$$

10.3 Theorem 3 could also have been formulated in terms of the formal integral defined by (2.6): $\tilde{Y}(z, u) = \tilde{\varphi}(z, u e^z) \in \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}[[u e^z]]$ and

$$\dot{\Delta}_m \tilde{Y} = C_m u^{m+1} \frac{\partial \tilde{Y}}{\partial u}, \quad m \in \mathbb{Z}^*,$$

where $\dot{\Delta}_m = e^{-mz} \Delta_m$ is the *dotted alien derivation of index m*, which already appeared in formula (9.24).

10.4 The Bridge Equations (10.3) are a compact writing of infinitely many “resurgence equations” for the series $\Delta_m \tilde{\varphi}_n$ or $\Delta_m \tilde{\psi}_n$, obtained by expanding them in powers of y .

For instance, setting

$$\tilde{\Phi}_n = \begin{cases} 1 + \tilde{\varphi}_1 & \text{if } n = 1 \\ \tilde{\varphi}_n & \text{if } n \neq 1, \end{cases} \quad (10.6)$$

so that $\tilde{\varphi}(z, y) = \sum_{n \geq 0} \tilde{\Phi}_n(z) y^n$, we get

$$\Delta_m \tilde{\Phi}_n = \begin{cases} (n-m) C_m \tilde{\Phi}_{n-m} & \text{if } -1 \leq m \leq n-1 \\ 0 & \text{if } m \leq -2 \text{ or } m \geq n. \end{cases}$$

Thus

- $\Delta_m \tilde{\varphi}_0 = 0$ for $m \neq -1$, while $\Delta_{-1} \tilde{\varphi}_0 = C_{-1}(1 + \tilde{\varphi}_1)$;
 - $\Delta_m \tilde{\varphi}_1 = 0$ for $m \neq -1$, while $\Delta_{-1} \tilde{\varphi}_1 = 2C_{-1} \tilde{\varphi}_2$;
 - $\Delta_m \tilde{\varphi}_2 = 0$ for $m \notin \{-1, 1\}$, while $\Delta_{-1} \tilde{\varphi}_2 = 3C_{-1} \tilde{\varphi}_3$ and $\Delta_1 \tilde{\varphi}_2 = C_1(1 + \tilde{\varphi}_1)$;
 - $\Delta_m \tilde{\varphi}_3 = 0$ for $m \notin \{-1, 1, 2\}$, while...
- \vdots

Similarly, with

$$\tilde{\Psi}_n = \begin{cases} 1 + \tilde{\psi}_1 & \text{if } n = 1 \\ \tilde{\psi}_n & \text{if } n \neq 1, \end{cases}$$

we have $\sum (\Delta_m \tilde{\Psi}_n) y^n = -C_m (\sum \tilde{\Psi}_n y^n)^{m+1}$, which means that $\Delta_m \tilde{\Psi}_n = 0$ for all $n \in \mathbb{N}$ when $m \leq -2$,

$$\Delta_{-1} \tilde{\Psi}_n = \begin{cases} -C_{-1} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

The second equation follows from (9.3) for the computation of $\Delta_m (\partial + y \frac{\partial}{\partial y}) \tilde{\varphi}(z, y)$, and from the relation $\Delta_m A(-1/z, \tilde{\varphi}(z, y)) = (\partial_y A(-1/z, \tilde{\varphi}(z, y))) \Delta_m \tilde{\varphi}(z, y)$ deduced from Proposition 10.2 below (indeed, $A(-1/z, y) \in \mathbb{C}\{z^{-1}, y\} \subset \mathbf{A}\{y\}$). Since $\partial_y \tilde{\varphi} = 1 + \mathcal{O}(z^{-1}, y)$ is invertible, we can set $\tilde{\chi} = (\partial_y \tilde{\varphi})^{-1} \Delta_m \tilde{\varphi}$; the above linear equations imply that $\tilde{\chi}$ is annihilated by $\tilde{X}_0 - (m+1)$, thus proportional to y^{m+1} : there exists $c_m \in \mathbb{C}$ such that $\Delta_m \tilde{\varphi} = c_m y^{m+1} \partial_y \tilde{\varphi}(z, y)$.

The relation $\Delta_m \tilde{\psi} = -c_m \tilde{\psi}^{m+1}$ follows by the alien chain rule: $y = \tilde{\varphi}(z, \tilde{\psi}(z, y)) = \tilde{\Theta}^{-1} \tilde{\varphi}$ implies $(\Delta_m \tilde{\varphi})(z, \tilde{\psi}) + \partial_y \tilde{\varphi}(z, \tilde{\psi}) \Delta_m \tilde{\psi} = 0$ by Proposition 10.2 below (using $\tilde{\varphi} \in \mathbf{A}\{y\}$).

and $\Delta_m \tilde{\Psi}_n = -C_m \sum_{n_1+\dots+n_{m+1}=n} \tilde{\Psi}_{n_1} \cdots \tilde{\Psi}_{n_{m+1}}$ for any $n \in \mathbb{N}$ and $m \geq 1$.

In particular, C_m is the constant term in $\Delta_m \tilde{\varphi}_{m+1}$ or in $-\Delta_m \tilde{\psi}_{m+1}$.

10.5 Proof of Proposition 10.1 and Theorem 3. We have a Fréchet space structure on $\widehat{H}(\mathcal{R}(\mathbb{Z}))$, with seminorms $\|\cdot\|_K$ indexed by the compact subsets of $\mathcal{R}(\mathbb{Z})$:

$$\|\widehat{\varphi}\|_K = \max_{\zeta \in K} |\widehat{\varphi}(\zeta)|, \quad \widehat{\varphi} \in \widehat{H}(\mathcal{R}(\mathbb{Z})), \quad K \in \mathcal{K}.$$

We thus naturally get Fréchet space structures on $\widehat{\mathbf{R}}_{\mathbb{Z}} = \mathbb{C}\delta \oplus \widehat{H}(\mathcal{R}(\mathbb{Z}))$, by defining $\|c\delta + \widehat{\varphi}\|_K := \max(|c|, \|\widehat{\varphi}\|_K)$, and on $\widetilde{\mathbf{R}}_{\mathbb{Z}} = \mathcal{B}^{-1}\widehat{\mathbf{R}}_{\mathbb{Z}}$, with $\|\widetilde{\chi}\|_K := \|\mathcal{B}\widetilde{\chi}\|_K$ for $\widetilde{\chi} = c + \widetilde{\varphi} \in \widetilde{\mathbf{R}}_{\mathbb{Z}}$.

The space $\mathbf{A} = \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ of simple resurgent functions is a closed subspace of $\widetilde{\mathbf{R}}_{\mathbb{Z}}$ and the Δ_m are continuous operators. Indeed, the map $\widehat{\varphi} \mapsto \text{sing}_m(\text{cont}_\gamma \widehat{\varphi})$ is continuous on $\widehat{\mathbf{A}} = \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ because the variation can be expressed as a difference of branches and the residuum as a Cauchy integral.

Consider now the formal series $\tilde{\mathcal{V}}^\omega(z) = \tilde{\mathcal{V}}_a^\omega(z)$, $\tilde{\mathcal{V}}^\omega(z) = \tilde{\mathcal{V}}_a^\omega(z) \in \mathbf{A}$, and their formal Borel transforms, which belong to $\widehat{\mathbf{A}}$. The end of the proof of Theorem 2 shows that $(\beta_\omega \tilde{\mathcal{V}}^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=n-1}$ and $(\beta_\omega \widehat{\mathcal{V}}^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=n-1}$ are summable families of $\widehat{\mathbf{A}}$ for each $n \in \mathbb{N}$; indeed, for any compact subset K of $\mathcal{R}(\mathbb{Z})$, there exist ρ, N and L such that any point of K is the endpoint of a $(\rho, N, n - \mathbb{N}^*)$ -adapted path of length $\leq L$ and also the endpoint of a (ρ, N, \mathbb{N}) -adapted path of length $\leq L$, and one can use (8.24), (8.27) and (8.28). Hence the sums $\widehat{\varphi}_n$ and $\widehat{\psi}_n$ of these families belong to $\widehat{\mathbf{A}}$. Equivalently, the formal series $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ appear as sums of summable families of \mathbf{A} :

$$\tilde{\varphi}_n = \sum_{\omega \in \Omega^\bullet, \|\omega\|=n-1} \beta_\omega \tilde{\mathcal{V}}^\omega \quad \text{and} \quad \tilde{\psi}_n = \sum_{\omega \in \Omega^\bullet, \|\omega\|=n-1} \beta_\omega \widehat{\mathcal{V}}^\omega \quad \text{in } \mathbf{A},$$

they are thus simple resurgent functions themselves. To end the proof of Theorem 3, we thus only have to study the alien derivatives $\Delta_m \tilde{\varphi}_n$ and $\Delta_m \tilde{\psi}_n$.

10.6 End of the proof of Proposition 10.1: Let $m \in \mathbb{Z}^*$. In view of Lemma 8.1, we can suppose $m \geq -1$. By continuity of Δ_m , $(\beta_\omega \Delta_m \tilde{\mathcal{V}}^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=m}$ is a summable family of \mathbf{A} , of sum $\Delta_m \tilde{\psi}_{m+1}$. In particular, the family obtained by extracting the constant terms is summable, but the constant term in $\Delta_m \tilde{\mathcal{V}}^\omega$ is $-V_a^\omega(m)$ by (9.10). Hence we get the summability of

$$C_m = \sum_{\|\omega\|=m} \beta_\omega V_a^\omega(m) \quad \text{in } \mathbb{C},$$

which is the constant term in $-\Delta_m \tilde{\psi}_{m+1}$. \square

10.7 As vector spaces, $\mathbb{C}[[y]]$ and $\mathbf{A}[[y]]$ can be identified with $\mathbb{C}^{\mathbb{N}}$ and $\mathbf{A}^{\mathbb{N}}$ and are thus also Fréchet spaces if we put the product topology on them.

As an intermediary step in the proof of Theorem 3, let us show

Lemma 10.1. *Let $m \in \mathbb{Z}^*$ and*

$$\mathcal{C}(m) = C_m y^{m+1} \frac{\partial}{\partial y}.$$

Then, for each $n_0 \in \mathbb{N}$, the families $(\tilde{\mathcal{V}}^\omega \mathbf{B}_\omega y^{n_0})_{\omega \in \Omega^\bullet}$ and $(V_a^\omega(m) \mathbf{B}_\omega y^{n_0})_{\omega \in \Omega^\bullet}$ are summable in $\mathbf{A}[[y]]$, of sums $\tilde{\Theta}^{-1} y^{n_0}$ and $\mathcal{C}(m) y^{n_0}$.

Proof. Our aim is to show that $(\tilde{\mathcal{V}}^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ and $(V_a^\omega(m) \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ are pointwise summable families of operators of $\mathbf{A}[[y]]$; in view of the above, since $\mathbf{B}_\omega y = \beta_\omega y^{\|\omega\|+1}$, we can already evaluate these operators on y and write

$$\sum_{\omega \in \Omega^\bullet} V_a^\omega(m) \mathbf{B}_\omega y = \sum_{\omega \in \Omega^\bullet, \|\omega\|=m} V_a^\omega(m) \mathbf{B}_\omega y = C_m y^{m+1} \quad \text{in } \mathbb{C}[[y]] \quad (10.7)$$

(the first identity stems from (9.11)) and

$$\sum_{\omega \in \Omega^\bullet} \tilde{\mathcal{V}}^\omega \mathbf{B}_\omega y = y + \sum_{n \geq 0} \tilde{\psi}_n(z) y^n = \tilde{\Theta}^{-1} y \quad \text{in } \mathbf{A}[[y]].$$

Although similar to formula (3.13), the last equation is stronger in that it gives the sum of a summable family of $\mathbf{A}[[y]]$ rather than of a formally summable family of $\mathbb{C}[[z^{-1}, y]]$.

When evaluating the operators \mathbf{B}_ω on y^{n_0} , we get coefficients β_{ω, n_0} which generalise the β_ω 's:

$$\mathbf{B}_\omega y^{n_0} = \beta_{\omega, n_0} y^{n_0 + \|\omega\|}$$

with $\beta_{\emptyset, n_0} = 1$, $\beta_{(\omega_1), n_0} = n_0$, $\beta_{\omega, n_0} = n_0(n_0 + \check{\omega}_1)(n_0 + \check{\omega}_2) \cdots (n_0 + \check{\omega}_{r-1})$ for $r \geq 2$. Notice that $\beta_{\omega, n_0} \neq 0 \Rightarrow \|\omega\| \geq -n_0$.

A suitable modification of the proof of Theorem 2 shows that the families $(\beta_{\omega, n_0} \tilde{\mathcal{V}}^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=m}$ are summable in $\hat{\mathbf{A}}$ for all $m \geq -n_0$ (replace the functions $S_m(\zeta) = \frac{m+1}{\zeta-m}$ of Lemma 8.2 by $\frac{m+n_0}{\zeta-m}$, for which the bounds are only slightly worse than in Lemma 8.4).

This yields the first part of the lemma, since we can now write

$$\sum_{\omega \in \Omega^\bullet} \tilde{\mathcal{V}}^\omega \mathbf{B}_\omega y^{n_0} = \sum_{m \geq -n_0} \left(\sum_{\omega \in \Omega^\bullet, \|\omega\|=m} \beta_{\omega, n_0} \tilde{\mathcal{V}}^\omega \right) y^{n_0+m} = \tilde{\Theta}^{-1} y^{n_0} \quad \text{in } \mathbf{A}[[y]].$$

By continuity of Δ_m , we also get the summability of $(\beta_{\omega, n_0} \Delta_m \tilde{\mathcal{V}}^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=m}$ in \mathbf{A} , hence of the family $(-\beta_{\omega, n_0} V_a^\omega(m))_{\omega \in \Omega^\bullet, \|\omega\|=m}$ obtained by extracting the constant terms. Let

$$C_{m, n_0} = \sum_{\omega \in \Omega^\bullet, \|\omega\|=m} \beta_{\omega, n_0} V_a^\omega(m) \quad \text{in } \mathbb{C}.$$

Thus $(V_a^\omega(m) \mathbf{B}_\omega y^{n_0})_{\omega \in \Omega^\bullet}$ is summable in $\mathbf{A}[[y]]$, with sum $C_{m, n_0} y^{n_0+m}$.

Let $\Omega^{k, R}$ ($k, R \in \mathbb{N}^*$) denote an exhaustion of Ω^\bullet by finite sets as in the proof of Proposition 6.1. We conclude by showing that $C_{m, n_0} y^{n_0+m} = \mathcal{C}(m) y^{n_0}$. This

follows from that fact that the operators

$$\mathcal{C}^{k,R}(m) = \sum_{\omega \in \Omega^{k,R}} V_a^\omega(m) \mathbf{B}_\omega$$

are all derivations of $\mathbb{C}[[y]]$ because of the alternality of $V_a^\bullet(m)$ (the Leibniz rule is easily checked with the help of the cosymmetry of \mathbf{B}_\bullet), thus their pointwise limit is also a derivation, which cannot be anything but $\mathcal{C}(m)$ by virtue of (10.7). \square

10.8 End of the proof of Theorem 3: In $\mathbf{A}[[y]]$, the families $(\tilde{\mathcal{V}}^\omega \mathbf{B}_\omega y)_{\omega \in \Omega^\bullet}$ and $(\tilde{\mathcal{V}}^\omega \mathbf{B}_\omega y)_{\omega \in \Omega^\bullet}$ are summable, of sums

$$\tilde{\varphi}(z, y) = \sum_{\omega \in \Omega^\bullet} \tilde{\mathcal{V}}^\omega(z) \mathbf{B}_\omega y, \quad \tilde{\psi}(z, y) = \sum_{\omega \in \Omega^\bullet} \tilde{\mathcal{V}}^\omega(z) \mathbf{B}_\omega y. \quad (10.8)$$

The derivation of $\mathbf{A}[[y]]$ induced by Δ_m is clearly continuous; applying Δ_m to both sides of the first equation in (10.8) and using (6.1) and (9.10), we find

$$\begin{aligned} \Delta_m \tilde{\varphi} &= \sum_{\omega} (\Delta_m \tilde{\mathcal{V}}^\omega) \mathbf{B}_\omega y = \sum_{\omega^1, \omega^2} \tilde{\mathcal{V}}^{\omega^1} V_a^{\omega^2}(m) \mathbf{B}_{\omega^2} \mathbf{B}_{\omega^1} y \\ &= \sum_{\omega^2} V_a^{\omega^2}(m) \mathbf{B}_{\omega^2} \tilde{\Theta} y = \mathcal{C}(m) \tilde{\varphi} \end{aligned}$$

(with the help of Lemma 10.1 for the last identities). Similarly,

$$\begin{aligned} \Delta_m \tilde{\psi} &= \sum_{\omega} (\Delta_m \tilde{\mathcal{V}}^\omega) \mathbf{B}_\omega y = - \sum_{\omega^1, \omega^2} V_a^{\omega^1}(m) \tilde{\mathcal{V}}^{\omega^2} \mathbf{B}_{\omega^2} \mathbf{B}_{\omega^1} y \\ &= - \sum_{\omega^2} \tilde{\mathcal{V}}^{\omega^2} \mathbf{B}_{\omega^2} \mathcal{C}(m) y = - \tilde{\Theta}^{-1}(C_m y^{m+1}) = - C_m (\tilde{\Theta}^{-1} y)^{m+1}. \end{aligned}$$

\square

10.9 Operator form of the Bridge Equation. As announced after the statement of Theorem 3, the Bridge Equation can be given a form which involves the operators $\tilde{\Theta}$ or $\tilde{\Theta}^{-1}$ in a more symmetric way. This will require a further construction.

Proposition 10.2. *Let $\mathbf{A} = \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$. The set*

$$\mathbf{A}\{y\} = \left\{ \sum_{n \geq 0} \tilde{f}_n(z) y^n \in \mathbf{A}[[y]] \mid \forall K \in \mathcal{K}, \exists c, \Lambda > 0 \text{ s.t. } \|\tilde{f}_n\|_K \leq c \Lambda^n \text{ for all } n \right\}$$

is a subalgebra of $\mathbf{A}[[y]]$, which contains $\tilde{\varphi}(z, y)$ and $\tilde{\psi}(z, y)$ and which is invariant by all the alien derivations Δ_m . Moreover, the substitution operators $\tilde{\Theta}$ and $\tilde{\Theta}^{-1}$ leave $\mathbf{A}\{y\}$ invariant and the operators they induce on $\mathbf{A}\{y\}$ satisfy the “alien chain rule”

$$\Delta_m \tilde{\Theta} \tilde{f} = \tilde{\Theta} \Delta_m \tilde{f} + (\tilde{\Theta} \partial_y \tilde{f}) \Delta_m \tilde{\varphi}, \quad \Delta_m \tilde{\Theta}^{-1} \tilde{f} = \tilde{\Theta}^{-1} \Delta_m \tilde{f} + (\tilde{\Theta}^{-1} \partial_y \tilde{f}) \Delta_m \tilde{\psi}.$$

Idea of the proof: The fact that $\tilde{\varphi}, \tilde{\psi} \in \mathbf{A}\{y\}$ follows easily from (8.3)–(8.4). The other statements require symmetrically contractile paths, first to control the seminorm $\|\cdot\|_K$ of a product of simple resurgent functions (\mathbf{A} is in fact a Fréchet algebra), and then to study $\partial_y^n \tilde{f}(z, \tilde{\varphi}_0(z))$ which appears in the substitution of $\tilde{\varphi}$ inside a series with resurgent coefficients:

$$\tilde{f}(z, \tilde{\varphi}) = \tilde{f}(z, \tilde{\varphi}_0) + y \partial_y \tilde{f}(z, \tilde{\varphi}_0) \tilde{\Phi}_1 + y^2 \left(\partial_y \tilde{f}(z, \tilde{\varphi}_0) \tilde{\Phi}_2 + \frac{1}{2!} \partial_y^2 \tilde{f}(z, \tilde{\varphi}_0) \tilde{\Phi}_1^2 \right) + \dots$$

with the notation (10.6). See [14] (e.g. §2.3, formula (41)). \square

Theorem 4. *We have the following identities in $\text{End}_{\mathbb{C}}(\mathbf{A}\{y\})$:*

$$[\Delta_m, \tilde{\Theta}] = \mathcal{C}(m)\tilde{\Theta}, \quad [\Delta_m, \tilde{\Theta}^{-1}] = -\tilde{\Theta}^{-1}\mathcal{C}(m), \quad (10.9)$$

for all $m \in \mathbb{Z}^*$.

Proof. We must prove that $\tilde{\Theta}\Delta_m\tilde{\Theta}^{-1} - \Delta_m = -\mathcal{C}(m)$.

The operators $\tilde{\Theta}$ and $\tilde{\Theta}^{-1}$ are mutually inverse \mathbf{A} -linear automorphisms of $\mathcal{A} = \mathbf{A}\{y\}$ and $\mathcal{C}(m)$ is an \mathbf{A} -linear derivation. The operator Δ_m is a derivation, it is not \mathbf{A} -linear, but $D = \tilde{\Theta}\Delta_m\tilde{\Theta}^{-1} - \Delta_m$ is an \mathbf{A} -linear derivation; indeed, if $\mu(z) \in \mathbf{A}$ and $f(z, y) \in \mathcal{A}$, then

$$\begin{aligned} D(\mu f) &= \tilde{\Theta}\Delta_m(\mu\tilde{\Theta}^{-1}f) - \Delta_m(\mu f) = \\ &= \tilde{\Theta}(\mu\Delta_m\tilde{\Theta}^{-1}f + (\Delta_m\mu)\tilde{\Theta}^{-1}f) - (\mu\Delta_m f + (\Delta_m\mu)f) = \\ &= \mu\tilde{\Theta}\Delta_m\tilde{\Theta}^{-1}f + (\Delta_m\mu)f - \mu\Delta_m f - (\Delta_m\mu)f = \mu Df. \end{aligned}$$

It is thus sufficient to check that the operator $D + \mathcal{C}(m)$ vanishes on y (being a continuous \mathbf{A} -linear derivation of \mathcal{A} , it'll have to vanish everywhere).

But, in view of (10.5), $Dy = \tilde{\Theta}\Delta_m\tilde{\psi} = -C_m(\tilde{\Theta}\tilde{\psi})^{m+1} = -C_my^{m+1}$, as required. \square

10.10 The Bridge Equation and the problem of analytic classification. We now explain why the coefficients C_m implied in the Bridge Equation are “analytic invariants” of the vector field X .

Suppose we are given two saddle-node vector fields, X_1 and X_2 , of the form (2.1) and satisfying (2.2). Both of them are formally conjugate to the normal form X_0 , hence they are mutually formally conjugate. Namely, we have formal substitution automorphisms Θ_i (or $\tilde{\Theta}_i$, when using the variable z instead of x) conjugating X_i with X_0 , for $i = 1, 2$, hence

$$\Theta X_1 = X_2 \Theta, \quad \Theta = \Theta_2^{-1} \Theta_1.$$

The operator Θ is the substitution operator associated with

$$\theta: (x, y) \mapsto (x, \varphi(x, y)), \quad \varphi(x, y) = \Theta y = \varphi_1(x, \psi_2(x, y)),$$

which is the unique formal transformation of the form (2.5) such that $X_1 = \theta^*X_2$.

One can check that, when passing to the variable z , one gets as a consequence of Proposition 10.2 and Theorem 4:

$$\tilde{\Theta} \in \text{End}_{\mathbb{C}}(\mathbf{A}\{y\}), \quad [\Delta_m, \tilde{\Theta}] = \tilde{\Theta}_2^{-1}(\mathcal{C}_1(m) - \mathcal{C}_2(m))\tilde{\Theta}_1, \quad m \in \mathbb{Z}^*,$$

where $\mathcal{C}_i(m) = C_{i,m}y^{m+1}\frac{\partial}{\partial y}$ is the derivation appearing in the right-hand side of the Bridge Equations (10.9) for X_i .

If X_1 and X_2 are holomorphically conjugate, then the unique formal conjugacy θ is given by a convergent series $\theta(x, y)$, thus all the alien derivatives of $\tilde{\varphi}$ vanish and $\mathcal{C}_1(m) = \mathcal{C}_2(m)$ for all m . We thus have proved half of

Theorem 5. *Two saddle-node vector fields of the form (2.1) and satisfying (2.2) are analytically conjugate if and only if their Bridge Equations (10.3) share the same collection of coefficients $(C_m)_{m \in \mathbb{Z}^*}$.*

According to this theorem, the numbers C_m thus constitute a complete system of analytic invariants for a saddle-node vector field.

To complete the proof of Theorem 5, one needs to show the reverse implication, i.e. that the identities $\mathcal{C}_1(m) = \mathcal{C}_2(m)$ imply the convergence of $\varphi_1(x, \psi_2(x, y))$. This will follow from the results of next section, according to which the coefficients C_m are related to another complete system of analytic invariants, which admits a more geometric description.

10.11 We end this section with a look at simple cases of the general theory.

“Euler equation” corresponds to $A(x, y) = x + y$, as mentioned in Section 2. We may call Euler-like equations those which correspond to the case in which $a_\eta = 0$ for $\eta \geq 1$, thus $A(x, y) = a_{-1}(x) + (1 + a_0(x))y$. For them, the formal integral is explicit.

Set $\tilde{a}_0(z) = a_0(-1/z) \in z^{-2}\mathbb{C}\{z^{-1}\}$ and $\tilde{a}_{-1}(z) = a_{-1}(-1/z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ as usual. Let $\tilde{\alpha}(z)$ be the unique series such that $\partial_z \tilde{\alpha} = \tilde{a}_0$ and $\tilde{\alpha} \in z^{-1}\mathbb{C}\{z^{-1}\}$. Set also $\tilde{\beta} = \tilde{a}_{-1}e^{-\tilde{\alpha}} \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $\hat{\beta} = \mathcal{B}\tilde{\beta}$ (which is an entire function of exponential type). One finds

$$\tilde{Y}(z, u) = \tilde{\varphi}_0(z) + u e^{z+\tilde{\alpha}(z)}, \quad \tilde{\varphi}_0 = -e^{\tilde{\alpha}} \mathcal{B}^{-1}\left(\zeta \mapsto \frac{\hat{\beta}(\zeta)}{\zeta + 1}\right).$$

Correspondingly, $\varphi(x, y) = \Phi_0(x) + \Phi_1(x)y$ with $\Phi_0(x) = \tilde{\varphi}_0(-1/x)$ generically divergent and $\Phi_1(x) = e^{\tilde{\alpha}(-1/x)}$ convergent.

One has $C_m = 0$ for every $m \in \mathbb{Z} \setminus \{-1\}$, but

$$C_{-1} = e^{-\tilde{\alpha}} \Delta_{-1} \tilde{\varphi}_0 = -2\pi i \hat{\beta}(-1).$$

10.12 Another particular case, much less trivial, is that of Riccati equations (see [4], [3, Vol. 2] or [1]): when $a_1 \neq 0$ and $a_\eta = 0$ for $\eta \geq 2$, hence $A(x, y) = a_{-1}(x) + (1 + a_0(x))y + a_1(x)y^2$, one can check that the formal integral has a linear fractional dependence upon the parameter u :

$$\tilde{Y}(z, u) = \frac{\tilde{\varphi}_0(z) + u e^z \tilde{\chi}(z)}{1 + u e^z \tilde{\chi}(z) \tilde{\varphi}_\infty(z)},$$

where $\tilde{\varphi}_0$, $\tilde{\varphi}_\infty$ and $-1 + \tilde{\chi}$ belong to $z^{-1}\mathbb{C}[[z^{-1}]]$; $\tilde{\varphi}_0$ and $1/\tilde{\varphi}_\infty$ can be found as the unique solutions of the differential equation (2.7) in the fraction field $\mathbb{C}((z^{-1}))$. Correspondingly, the normalising series $\varphi(x, y)$ and $\psi(x, y)$ have a linear fractional dependence upon y .

In the Riccati case, only C_{-1} and C_1 may be nonzero. Indeed,

$$\Delta_m \tilde{\varphi}_0 \neq 0 \Rightarrow m = -1, \quad \Delta_m \tilde{\varphi}_\infty \neq 0 \Rightarrow m = 1, \quad \Delta_m \tilde{\chi} \neq 0 \Rightarrow m = \pm 1.$$

10.13 We may call “canonical Riccati equations” the equations corresponding to a function A of the form $A(x, y) = y + \frac{1}{2\pi i}B_-x + \frac{1}{2\pi i}B_+xy^2$, with $B_-, B_+ \in \mathbb{C}$. Thus, for them, the differential equation (2.7) reads

$$\partial_z \tilde{Y} = \tilde{Y} - \frac{1}{2\pi iz}(B_- + B_+\tilde{Y}^2).$$

A direct mould computation based on (10.1) is given in [3], Vol. 2, pp. 476–480, yielding

$$C_{-1} = B_- \sigma(B_- B_+), \quad C_1 = -B_+ \sigma(B_- B_+),$$

with $\sigma(b) = \frac{2}{b^{1/2}} \sin \frac{b^{1/2}}{2}$ (see [1] for a computation by another method).

11 Relation with Martinet-Ramis's invariants

In this section, we continue to investigate the consequences of the resurgence of the solution of the conjugacy equation for a saddle-node X . We shall now connect the “alien computations” of the previous section with Martinet-Ramis’s solution of the problem of analytic classification [12], completing at the same time the proof of Theorem 5.

This will be done by comparing sectorial solutions of the conjugacy problem obtained by Borel-Laplace summation on the one hand, and by deriving geometric consequences of the Bridge Equation through exponentiation and summation on the other hand (this amounts to a resurgent description of the “Stokes phenomenon” for the differential equation (2.7)).

11.1 Let us call *Martinet-Ramis’s invariants* of X the numbers $\xi_{-1}, \xi_1, \xi_2, \dots$ defined in terms of Écalle’s invariants by the formulas

$$\xi_{-1} = -C_{-1}, \tag{11.1}$$

$$\xi_m = \sum_{r \geq 1} \sum_{\substack{m_1, \dots, m_r \geq 1 \\ m_1 + \dots + m_r = m}} \frac{(-1)^r}{r!} \beta_{m_1, \dots, m_r} C_{m_1} \dots C_{m_r}, \quad m \geq 1, \tag{11.2}$$

where, as usual, $\beta_{m_1} = 1$ and $\beta_{m_1, \dots, m_r} = (m_1+1)(m_1+m_2+1) \cdots (m_1+\dots+m_{r-1})$ for $r \geq 2$.

Observe that they are obtained by integrating backwards the vector fields

$$\mathcal{C}_- = \mathcal{C}(-1) = C_{-1} \frac{\partial}{\partial u}, \quad \mathcal{C}_+ = \sum_{m>0} \mathcal{C}(m) = \sum_{m>0} C_m u^{m+1} \frac{\partial}{\partial u}.$$

Indeed, the time- (-1) maps of \mathcal{C}_- and \mathcal{C}_+ are

$$u \mapsto \xi_-(u) = u + \xi_{-1}, \quad u \mapsto \xi_+(u) = u + \sum_{m>0} \xi_m u^{m+1} \quad (11.3)$$

(as can be checked by viewing $-\mathcal{C}_+$ as an elementary mould-comould expansion on the alphabet \mathbb{N}^* ; the reason for changing the variable y into u will appear later).¹⁴

These numbers can also be defined directly from the iterated integrals L_a^ω of (9.18):

Proposition 11.1. *The family $(\beta_\omega L_a^\omega)_{\omega \in \Omega^\bullet, \|\omega\|=m}$ is summable in \mathbb{C} for each $m \in \mathbb{Z}^*$ and*

$$\xi_m = \sum_{\omega \in \Omega^\bullet, \|\omega\|=m} \beta_\omega L_a^\omega,$$

with the convention $\xi_m = 0$ for $m \leq -2$.

Idea of the proof. The relations $\xi_\pm(u) = \sum_{\omega \in \Omega^\bullet} L_{a,\pm}^\omega \mathbf{B}_\omega u$ (where $L_{a,\pm}^\omega$ is defined by (9.21)) formally follow from the formula $L_{a,\pm}^\bullet = \exp(-V_{a,\pm}^\bullet)$ and Lemma 10.1, according to which $(V_{a,\pm}^\omega \mathbf{B}_\omega)_{\omega \in \Omega^\bullet}$ is a pointwise summable family of operators of $\mathbf{A}[[u]]$ with sum \mathcal{C}_\pm . The summability can be justified by the same kind of arguments as in the proof of Proposition 10.1 and Theorem 3. \square

11.2 The formulas (11.1)–(11.2) can be inverted so as to express the C_m 's in terms of the ξ_m 's. Theorem 5 is thus equivalent to the fact that the ξ_m 's constitute themselves a complete system of analytic invariants for the saddle-node classification problem. We shall now prove this fact directly.

In fact, we shall obtain more: the pair (ξ_-, ξ_+) is a complete system of analytic invariants and ξ_+ is necessarily convergent. Thus, not all collections of numbers $(C_m)_{m \in \{-1\} \cup \mathbb{N}^*}$ can appear as analytic invariants, only those for which the corresponding ξ_m 's admit geometric bounds $|\xi_m| \leq K^m$ for $m \geq 1$ (hence they have to satisfy Gevrey bounds themselves: $|C_m| \leq K_1^m m!$ for $m \geq 1$).

This information will follow from the geometric interpretation of ξ_\pm . Martinet and Ramis have also showed that any collection $(\xi_m)_{m \in \{-1\} \cup \mathbb{N}^*}$ subject to the previous growth constraint can be obtained as a system of analytic invariants for some saddle-node vector field, but we shall not consider this question here.

11.3 Let us consider the saddle-node vector field X and its normal form X_0 in the variable $z = -1/x$ instead of x :

$$\tilde{X} = \frac{\partial}{\partial z} + A(-1/z, y) \frac{\partial}{\partial y}, \quad \tilde{X}_0 = \frac{\partial}{\partial z} + y \frac{\partial}{\partial y}.$$

For $\varepsilon \in]0, \pi/2[$ and $R > 0$, we set

$$\begin{aligned} \mathcal{D}^{up}(R, \varepsilon) &= \{ z \in \mathbb{C} \mid -\frac{\pi}{2} + \varepsilon \leq \arg z \leq \frac{3\pi}{2} - \varepsilon, |z| \geq R \}, \\ \mathcal{D}^{low}(R, \varepsilon) &= \{ z \in \mathbb{C} \mid -\frac{3\pi}{2} + \varepsilon \leq \arg z \leq \frac{\pi}{2} - \varepsilon, |z| \geq R \}, \end{aligned}$$

¹⁴Thus one always has $\xi_-(u) = u - C_{-1}$, and in the Riccati case as at the end of the previous section $\xi_+(u) = \frac{u}{1-C_1 u}$.

which are “sectorial neighbourhoods of infinity” in the z -plane (corresponding to certain sectorial neighbourhoods of the origin in the x -plane). Their intersection has two connected components:

$$\begin{aligned}\mathcal{D}_-(R, \varepsilon) &= \{ z \in \mathbb{C} \mid \frac{\pi}{2} + \varepsilon \leq \arg z \leq \frac{3\pi}{2} - \varepsilon, |z| \geq R \} \subset \{ \Re z < 0 \}, \\ \mathcal{D}_+(R, \varepsilon) &= \{ z \in \mathbb{C} \mid -\frac{\pi}{2} + \varepsilon \leq \arg z \leq \frac{\pi}{2} - \varepsilon, |z| \geq R \} \subset \{ \Re z > 0 \}.\end{aligned}$$

Theorem 6. *Let $\varepsilon \in]0, \pi/2[$. Then there exist $R, \rho > 0$ such that:*

- (i) *By Borel-Laplace summation, the formal series $\tilde{\varphi}_n(z)$ give rise to functions $\tilde{\varphi}_n^{up}(z)$, resp. $\tilde{\varphi}_n^{low}(z)$, which are analytic in $\mathcal{D}^{up}(R, \varepsilon)$, resp. $\mathcal{D}^{low}(R, \varepsilon)$, such that the formulas*

$$\tilde{\varphi}^{up}(z, y) = \sum_{n \geq 0} \tilde{\varphi}_n^{up}(z) y^n, \quad \tilde{\varphi}^{low}(z, y) = \sum_{n \geq 0} \tilde{\varphi}_n^{low}(z) y^n$$

define two functions $\tilde{\varphi}^{up}$ and $\tilde{\varphi}^{low}$ analytic in $\mathcal{D}^{up}(R, \varepsilon) \times \{ |y| \leq \rho \}$, resp. $\mathcal{D}^{low}(R, \varepsilon) \times \{ |y| \leq \rho \}$, and each of the transformations

$$\tilde{\theta}^{up}(z, y) = (z, \tilde{\varphi}^{up}(z, y)), \quad \tilde{\theta}^{low}(z, y) = (z, \tilde{\varphi}^{low}(z, y))$$

is injective in its domain and establishes there a conjugacy between the normal form \tilde{X}_0 and the saddle-node vector field \tilde{X} .

- (ii) *The series ξ_+ of (11.3) has positive radius of convergence and the upper and lower normalisations are connected by the formulas*

$$\tilde{\theta}^{up}(z, y) = \tilde{\theta}^{low}(z, \xi_-(y e^{-z}) e^z) = \tilde{\theta}^{low}(z, y + \xi_{-1} e^z)$$

for $z \in \mathcal{D}_-(R, \varepsilon)$ and $|y| \leq \rho$, whereas

$$\tilde{\theta}^{low}(z, y) = \tilde{\theta}^{up}(z, \xi_+(y e^{-z}) e^z) = \tilde{\theta}^{up}(z, y + \xi_1 y^2 e^{-z} + \xi_2 y^3 e^{-2z} + \dots)$$

for $z \in \mathcal{D}_+(R, \varepsilon)$ and $|y| \leq \rho$.

- (iii) *The pair (ξ_-, ξ_+) is a complete system of analytic invariants for X .*

As already mentioned, Theorem 6 contains Theorem 5. The rest of this section is devoted to the proof of Theorem 6.

11.4 In view of inequalities (8.4)–(8.5), the principal branches of the Borel transforms $\widehat{\varphi}_n(\zeta)$ and $\widehat{\psi}_n(\zeta)$ admit exponential bounds of the form $KL^n e^{C|\zeta|}$ in the sectors $\{ \zeta \in \mathbb{C} \mid \frac{\varepsilon}{2} \leq \arg \zeta \leq \pi - \frac{\varepsilon}{2} \}$ and $\{ \zeta \in \mathbb{C} \mid \pi + \frac{\varepsilon}{2} \leq \arg \zeta \leq 2\pi - \frac{\varepsilon}{2} \}$. Using the directions of the first sector for instance, we can define analytic functions by gluing the Laplace transforms corresponding to various directions

$$\tilde{\varphi}_n^{low}(z) = \int_0^{e^{i\theta}\infty} \widehat{\varphi}_n(\zeta) e^{-z\zeta} d\zeta, \quad \tilde{\psi}_n^{low}(z) = \int_0^{e^{i\theta}\infty} \widehat{\psi}_n(\zeta) e^{-z\zeta} d\zeta,$$

with $\theta \in [\frac{\varepsilon}{2}, \pi - \frac{\varepsilon}{2}]$. If we take R large enough, then the union of the half-planes $\{ \operatorname{Re}(z e^{i\theta}) > C \}$ contains $\mathcal{D}^{\text{low}}(R, \varepsilon)$ and the functions

$$\tilde{\varphi}^{\text{low}}(z, y) = \sum_{n \geq 0} \tilde{\varphi}_n^{\text{low}}(z) y^n, \quad \tilde{\psi}^{\text{low}}(z, y) = \sum_{n \geq 0} \tilde{\psi}_n^{\text{low}}(z) y^n$$

are analytic for $z \in \mathcal{D}^{\text{low}}(R, \varepsilon)$ and $|y| \leq \rho$ as soon as $\rho < 1/L$.

The standard properties of Borel-Laplace summation ensure that the relations $y = \tilde{\varphi}(z, \tilde{\psi}(z, y)) = \psi(z, \tilde{\varphi}(z, y))$ and $\tilde{X}_0 \tilde{\varphi}(z, y) = A(-1/z, \tilde{\varphi}(z, y))$ yield similar relations for $\tilde{\varphi}^{\text{low}}$ and $\tilde{\psi}^{\text{low}}$, possibly in smaller domains (because $\tilde{\varphi}^{\text{low}}(z, y) - y$ and $\tilde{\psi}^{\text{low}}(z, y) - y$ can be made uniformly small by increasing R and diminishing ρ). Hence the transformations

$$(z, y) \mapsto (z, \tilde{\varphi}^{\text{low}}(z, y)), \quad (z, y) \mapsto (z, \tilde{\psi}^{\text{low}}(z, y))$$

(or rather the sectorial germs they represent) are mutually inverse and establish a conjugacy between \tilde{X}_0 and \tilde{X} .

We define similarly $\tilde{\varphi}^{\text{up}}(z, y)$ and $\tilde{\psi}^{\text{up}}(z, y)$ with the desired properties, by means of Laplace transforms in directions belonging to $[\pi + \frac{\varepsilon}{2}, 2\pi - \frac{\varepsilon}{2}]$. This yields the first statement in Theorem 6.

11.5 We now have at our disposal two sectorial normalisations

$$\tilde{\theta}^{\text{low}} : (z, y) \mapsto (z, \tilde{\varphi}^{\text{low}}(z, y)), \quad \tilde{\theta}^{\text{up}} : (z, y) \mapsto (z, \tilde{\varphi}^{\text{up}}(z, y)),$$

which are defined in different but overlapping domains, and which admit the same asymptotic expansion with respect to z (when one first expands in powers of y). If we consider $(\tilde{\theta}^{\text{up}})^{-1} \circ \tilde{\theta}^{\text{low}}$ or $(\tilde{\theta}^{\text{low}})^{-1} \circ \tilde{\theta}^{\text{up}}$ in one of the two components of $\mathcal{D}^{\text{low}}(R, \varepsilon) \cap \mathcal{D}^{\text{up}}(R, \varepsilon)$, we thus get a transformation of the form

$$(z, y) \mapsto (z, \chi(z, y)) \tag{11.4}$$

which conjugates the normal form \tilde{X}_0 with itself, to which one can apply the following:

Lemma 11.1. *Let \mathcal{D} be a domain in \mathbb{C} . Suppose that the transformation $(z, y) \mapsto (z, \chi(z, y))$ is analytic and injective for $z \in \mathcal{D}$ and $|y| \leq \rho$, and that it conjugates \tilde{X}_0 with itself. Then there exists $\xi(u) \in \mathbb{C}\{u\}$ such that*

$$\chi(z, y) = \xi(y e^{-z}) e^z. \tag{11.5}$$

Such transformations are called *sectorial isotropies* of the normal form.

Proof. By assumption $\chi = \tilde{X}_0 \chi$. Since $y = \tilde{X}_0 y$, this implies that $\frac{1}{y} \chi(z, y)$ is a first integral of \tilde{X}_0 . Thus $\frac{1}{u e^z} \chi(z, u e^z)$ is independent of z and can be written $\underline{\xi(u)}$, where obviously $\xi(u) \in \mathbb{C}\{u\}$. \square

When $\chi(z, y)$ comes from $(\tilde{\theta}^{\text{up}})^{-1} \circ \tilde{\theta}^{\text{low}}$ or $(\tilde{\theta}^{\text{low}})^{-1} \circ \tilde{\theta}^{\text{up}}$, we have a further piece of information: in the Taylor expansion $\chi(z, y) - y = \sum_{n \geq 0} \chi_n(z) y^n$, each component $\chi_n(z)$ admits the null series as asymptotic expansion in $\mathcal{D}_{\pm}(R, \varepsilon)$ (the

transformation (11.4) is asymptotic to the identity because $\tilde{\theta}^{up}$ and $\tilde{\theta}^{low}$ share the same asymptotic expansion). This has different implications according to whether the domain \mathcal{D} is $\mathcal{D}_-(R, \varepsilon)$ or $\mathcal{D}_+(R, \varepsilon)$.

Indeed, if we expand $\xi(u) - u = \sum_{n \geq 0} \alpha_n u^n$, we get

$$\chi_0(z) = \alpha_0 e^z, \quad \chi_1(z) = \alpha_1, \quad \chi_2(z) = \alpha_2 e^{-z}, \quad \chi_3(z) = \alpha_3 e^{-2z}, \dots$$

hence

$$\begin{aligned} \mathcal{D} = \mathcal{D}_-(R, \varepsilon) &\subset \{ \Re z < 0 \} \Rightarrow \alpha_n = 0 \text{ for } n \neq 0, \\ \mathcal{D} = \mathcal{D}_+(R, \varepsilon) &\subset \{ \Re z > 0 \} \Rightarrow \alpha_0 = \alpha_1 = 0. \end{aligned}$$

The upshot is that there exist $\alpha_0 \in \mathbb{C}$ and $\xi(u) = u + \alpha_2 u^2 + \alpha_3 u^3 \in \mathbb{C}\{u\}$ such that

$$\begin{aligned} (\tilde{\theta}^{low})^{-1} \circ \tilde{\theta}^{up}(z, y) &= (z, y + \alpha_0 e^z), & z \in \mathcal{D}_-(R, \varepsilon), \\ (\tilde{\theta}^{up})^{-1} \circ \tilde{\theta}^{low}(z, y) &= (z, y + \alpha_2 y^2 e^{-z} + \alpha_3 y^3 e^{-2z} + \dots), & z \in \mathcal{D}_+(R, \varepsilon). \end{aligned}$$

11.6 It is elementary to check that the pair of sectorial isotopies $((\tilde{\theta}^{low})^{-1} \circ \tilde{\theta}^{up})|_{\mathcal{D}_-(R, \varepsilon)}, ((\tilde{\theta}^{up})^{-1} \circ \tilde{\theta}^{low})|_{\mathcal{D}_+(R, \varepsilon)})$ is a complete system of analytic invariants for X : suppose indeed that two saddle-node vector fields X_1 and X_2 are given and that we wish to know whether the unique formal transformation θ of the form (2.5) which conjugate them is convergent, then $\tilde{\theta}_2^{up} \circ (\tilde{\theta}_1^{up})^{-1}$ and $\tilde{\theta}_2^{low} \circ (\tilde{\theta}_1^{low})^{-1}$ are two sectorial conjugacies between X_1 and X_2 defined in different but overlapping domains and admitting θ as asymptotic expansion (up to the change $x = -1/z$); they coincide and define an analytic conjugacy iff $(\tilde{\theta}_2^{low})^{-1} \circ \tilde{\theta}_2^{up} = (\tilde{\theta}_1^{low})^{-1} \circ \tilde{\theta}_1^{up}$ in both components of the intersection of the domains.

11.7 Therefore, it only remains to be checked that $\alpha_0 = \xi_1$ and $\xi = \xi_+$. This will follow from the interpretation of the operators Δ_m^+ as components of the ‘‘Stokes automorphism’’. For this part, the reader may consult the end of §2.4 in [14].

Suppose that a simple resurgent functions $c\delta + \widehat{\varphi} \in \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ has the following property: the functions $\widehat{\chi}_m$ defined by $\Delta_m^+(c\delta + \widehat{\varphi}) = \gamma_m \delta + \widehat{\chi}_m$ and $\widehat{\varphi}$ itself have at most exponential growth in each non-horizontal directions, so that one can consider the Laplace transforms $\mathcal{L}^\theta \widehat{\varphi}(z) = \int_0^{e^{i\theta} \infty} \widehat{\varphi}(\zeta) e^{-z\zeta} d\zeta$ or $\mathcal{L}^\theta \widehat{\chi}_m(z)$ for $\theta \in]\varepsilon, \pi - \varepsilon[$ or $\theta \in]\pi + \varepsilon, 2\pi - \varepsilon[$, which are analytic in sectorial neighbourhoods of infinity of the form $\mathcal{D}^{low}(R, \varepsilon)$ or $\mathcal{D}^{up}(R, \varepsilon)$. Let $\theta < 0 < \theta'$, with θ and θ' both close to 0; by deforming a contour of integration, one deduces from the definition (9.19) that, for any $M \in \mathbb{N}^*$ and $\sigma \in]0, 1[$,

$$c + \mathcal{L}^\theta \widehat{\varphi}(z) = c + \mathcal{L}^{\theta'} \widehat{\varphi}(z) + \sum_{m=1}^M e^{-mz} (\gamma_m + \mathcal{L}^{\theta'} \widehat{\chi}_m(z)) + O(|e^{-(M+\sigma)z}|)$$

in the sectorial neighbourhood of infinity obtained by imposing that both $\Re(z e^{i\theta})$ and $\Re(z e^{i\theta'})$ be large enough, which is contained in the right half-plane $\{ \Re z > 0 \}$.

Let us denote this by: $\mathcal{L}^\theta(c\delta + \hat{\varphi}) \sim \sum_{m \geq 0} e^{-mz} \mathcal{L}^{\theta'} \Delta_m^+(c\delta + \hat{\varphi})$ in $\{\Re z > 0\}$. Similarly, if $\theta < \pi < \theta'$ with θ and θ' both close to π , one gets $\mathcal{L}^\theta(c\delta + \hat{\varphi}) \sim \sum_{m \leq 0} e^{-mz} \mathcal{L}^{\theta'} \Delta_m^+(c\delta + \hat{\varphi})$ in the left half-plane $\{\Re z < 0\}$.

We can even write $\mathcal{L}^\theta \sim \mathcal{L}^{\theta'} \circ \sum_{m \geq 0} \dot{\Delta}_m^+$ in $\{\Re z > 0\}$ and $\mathcal{L}^\theta \sim \mathcal{L}^{\theta'} \circ \sum_{m \leq 0} \dot{\Delta}_m^+$ in $\{\Re z < 0\}$, if we define properly $\dot{\Delta}_m^+$ in the convolutive model. See [14]: $\dot{\Delta}_m^+ = \tau_m \circ \Delta_m^+$, with a shift operator $\tau_m: \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}} \rightarrow \tau_m(\widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}})$, the target space being the set of simple resurgent functions “based at m ” (instead of being based at the origin). On the other hand, we can rephrase (9.24) as $\sum_{m \geq 0} \dot{\Delta}_m^+ = \exp\left(\sum_{m > 0} \dot{\Delta}_m^+\right)$, $\sum_{m \leq 0} \dot{\Delta}_m^+ = \exp\left(\sum_{m < 0} \dot{\Delta}_m^+\right)$.

Apply this to $\tilde{Y}(z, u)$ (or, rather, to each of its components): when θ and θ' are close to 0, we have $\mathcal{L}^\theta \tilde{Y} = \tilde{Y}^{up}$ and $\mathcal{L}^{\theta'} \tilde{Y} = \tilde{Y}^{low}$ in $\mathcal{D}_+(R, \varepsilon)$, hence, in view of the Bridge Equation, $\tilde{Y}^{up} \sim (\mathcal{L}^{\theta'} \circ \exp \mathcal{C}_+) \tilde{Y}$, which yields $\tilde{Y}^{up}(z, u) \sim \tilde{Y}^{low}(z, (\xi_+)^{-1}(u))$ in $\mathcal{D}_+(R, \varepsilon)$. Similarly, $\tilde{Y}^{low}(z, u) \sim \tilde{Y}^{up}(z, (\xi_-)^{-1}(u))$ in the domain $\mathcal{D}_-(R, \varepsilon)$. When interpreting these relations componentwise with respect to u and modulo $O(|e^{\pm(M+\sigma)z}|)$ in $\mathcal{D}_{\pm}(R, \varepsilon)$ with arbitrarily large M , we get the desired relations between $\tilde{\varphi}^{up}(z, y) = \tilde{Y}^{up}(z, ye^{-z})$ and $\tilde{\varphi}^{low}(z, y) = \tilde{Y}^{low}(z, ye^{-z})$.

12 The resurgence monomials $\tilde{\mathcal{U}}_a^\omega$'s and the freeness of alien derivations

12.1 The first goal of this section is to construct families of simple resurgent functions which form closed systems for multiplication and alien derivations in the following sense:

Definition 12.1. We call Δ -friendly monomials the members of any family of simple resurgent functions $(\tilde{\mathcal{U}}^{\omega_1, \dots, \omega_r})_{r \geq 0, \omega_1, \dots, \omega_r \in \mathbb{Z}^*}$, such that on the one hand

$$\Delta_m \tilde{\mathcal{U}}^{\omega_1, \dots, \omega_r} = \begin{cases} \tilde{\mathcal{U}}^{\omega_2, \dots, \omega_r} & \text{if } r \geq 1 \text{ and } \omega_1 = m, \\ 0 & \text{if not,} \end{cases} \quad (12.1)$$

for every $m \in \mathbb{Z}^*$, and on the other hand $\tilde{\mathcal{U}}^\emptyset = 1$ and

$$\tilde{\mathcal{U}}^\alpha \tilde{\mathcal{U}}^\beta = \sum_{\omega \in \Omega^\bullet} \text{sh} \binom{\alpha, \beta}{\omega} \tilde{\mathcal{U}}^\omega, \quad \alpha, \beta \in (\mathbb{Z}^*)^\bullet,$$

i.e., when viewed as a mould, $\tilde{\mathcal{U}}^\bullet \in \mathcal{M}^\bullet(\mathbb{Z}^*, \widehat{\text{RES}}_{\mathbb{Z}}^{\text{simp}})$ is symmetral.

J. Écalle calls Δ -friendly such resurgent functions by contrast with the functions $\tilde{\mathcal{V}}_a^{\omega_1, \dots, \omega_r}$, which can be termed “ ∂ -friendly monomials” because of (9.4) (using ∂ as short-hand for $\frac{d}{dz}$).

As a matter of fact, Δ -friendly monomials will be defined with the help of the moulds \tilde{V}_a^\bullet , V_a^\bullet of Section 9 and mould composition, but we first need to enlarge slightly the definition of mould composition.

12.2 We thus begin with a kind of addendum to Sections 4 and 5. Assume that \mathbf{A} is a commutative \mathbb{C} -algebra, the unit of which is denoted 1, and Ω is a commutative semigroup, the operation of which is denoted additively. We still use the notations $\|\omega\| = \omega_1 + \dots + \omega_r$ and $\|\emptyset\| = 0$.

Let us call *restricted moulds* the elements of $\mathcal{M}^\bullet(\Omega^*, \mathbf{A})$, where $\Omega^* = \Omega \setminus \{0\}$. The example we have in mind is $\Omega = \mathbb{Z}$ and $\mathbf{A} = \mathbb{C}$ or $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$.

Definition 12.2. We call *licit mould* any restricted mould U^\bullet such that

$$\|\omega\| = 0 \Rightarrow U^\omega = 0$$

for any $\omega \in (\Omega^*)^\bullet$. The set of licit moulds will be denoted $\mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A})$.

The set $\mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A})$ is clearly an \mathbf{A} -submodule of $\mathcal{M}^\bullet(\Omega^*, \mathbf{A})$, but not an \mathbf{A} -subalgebra. Notice that $U^\bullet \in \mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A})$ implies $U^\emptyset = 0$.

We now define the *composition of a restricted mould and a licit mould* as follows:

$$(M^\bullet, U^\bullet) \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A}) \times \mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A}) \mapsto C^\bullet = M^\bullet \circ U^\bullet \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A}),$$

with $C^\emptyset = M^\emptyset$ and, for $\omega \neq \emptyset$,

$$C^\omega = \sum_{\substack{s \geq 1, \omega = \omega^1 \dots \omega^s \\ \|\omega^1\|, \dots, \|\omega^s\| \neq 0}} M^{(\|\omega^1\|, \dots, \|\omega^s\|)} U^{\omega^1} \dots U^{\omega^s}.$$

The map $M^\bullet \mapsto M^\bullet \circ U^\bullet$ is clearly \mathbf{A} -linear; we leave it to the reader¹⁵ to check that it is an \mathbf{A} -algebra homomorphism, that

$$U^\bullet, V^\bullet \in \mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A}) \Rightarrow U^\bullet \circ V^\bullet \in \mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A}),$$

and that

$$M^\bullet \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A}) \text{ and } U^\bullet, V^\bullet \in \mathcal{M}_{lic}^\bullet(\Omega^*, \mathbf{A}) \Rightarrow (M^\bullet \circ U^\bullet) \circ V^\bullet = M^\bullet \circ (U^\bullet \circ V^\bullet).$$

The *restricted identity mould* is

$$I_*^\bullet: \omega \in (\Omega^*)^\bullet \mapsto I_*^\omega = \begin{cases} 1 & \text{if } r(\omega) = 1, \\ 0 & \text{if } r(\omega) \neq 1. \end{cases}$$

It is a licit mould, which satisfies $M^\bullet \circ I_*^\bullet = M^\bullet$ for any restricted mould M^\bullet and $I_*^\bullet \circ U^\bullet = U^\bullet$ for any licit mould U^\bullet . One can check that a licit mould U^\bullet admits an inverse for composition iff U^ω is invertible in \mathbf{A} whenever $r(\omega) = 1$.

A proposition analogous to Proposition 5.3 holds. In particular, *alternating invertible licit moulds form a subgroup of the composition group of invertible licit*

¹⁵The verification of most of the properties indicated in this paragraph can be simplified by observing that the canonical restriction map $\rho: \mathcal{M}^\bullet(\Omega, \mathbf{A}) \rightarrow \mathcal{M}^\bullet(\Omega^*, \mathbf{A})$ is an \mathbf{A} -algebra homomorphism which satisfies $\rho(M^\bullet \circ U^\bullet) = \rho(M^\bullet) \circ \rho(U^\bullet)$ for any two moulds M^\bullet and U^\bullet such that $\rho(U^\bullet)$ is licit and which preserves alternality and symmetrality.

*moulds.*¹⁶ The property that inversion of licit moulds preserves alternality will be used in the next paragraph.

12.3 We now take $\Omega = \mathbb{Z}$ and $\mathbf{A} = \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$. Assume that $a = (\widehat{a}_\eta)_{\eta \in \mathbb{Z}^*}$ is any family of entire functions such that $\widehat{a}_\eta(\eta) \neq 0$ for each $\eta \in \mathbb{Z}^*$. We still use the notations $\tilde{a}_\eta = \mathcal{B}^{-1}\widehat{a}_\eta \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $J_a^\omega = \tilde{a}_\eta$ if $\omega = (\eta)$, 0 if not. We recall that, according to Section 9, the equation

$$(\partial + \nabla)\tilde{\mathcal{V}}_a^\bullet = -\tilde{\mathcal{V}}_a^\bullet \times J_a^\bullet \quad (12.2)$$

defines a symmetral mould $\tilde{\mathcal{V}}_a^\bullet \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A})$, and that, for each $m \in \mathbb{Z}^*$, we have an alternal scalar mould $V_a^\bullet(m) = -V_a^\bullet(m) \in \mathcal{M}^\bullet(\Omega^*, \mathbb{C})$ which satisfies

$$\Delta_m \tilde{\mathcal{V}}_a^\bullet = V_a^\bullet(m) \times \tilde{\mathcal{V}}_a^\bullet, \quad (12.3)$$

$$V_a^\omega(m) \neq 0 \Rightarrow \|\omega\| = m. \quad (12.4)$$

Moreover $V_a^{(\eta)}(\eta) = 2\pi i \widehat{a}_\eta(\eta)$.

Theorem 7. *The formula $V_a^\bullet = \sum_{m \in \mathbb{Z}^*} V_a^\bullet(m)$ defines an alternal scalar licit mould, which admits a composition inverse U_a^\bullet . The formula*

$$\tilde{\mathcal{U}}_a^\bullet = \tilde{\mathcal{V}}_a^\bullet \circ U_a^\bullet \in \mathcal{M}^\bullet(\Omega^*, \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}) \quad (12.5)$$

defines a family of Δ -friendly monomials $\tilde{\mathcal{U}}_a^\omega$.

Proof. In view of (12.4), the definition of V_a^\bullet makes sense and its alternality follows from the alternality of each $V_a^\bullet(m)$. This mould is clearly licit, and $V_a^{(\eta)} = 2\pi i \widehat{a}_\eta(\eta) \neq 0$, hence its invertibility.

The general properties of the composition of a restricted mould and a licit mould ensure that (12.5) defines a symmetral mould. Its alien derivatives are easily computed since U_a^\bullet is a scalar mould:

$$\Delta_m \tilde{\mathcal{U}}_a^\bullet = (\Delta_m \tilde{\mathcal{V}}_a^\bullet) \circ U_a^\bullet = (V_a^\bullet(m) \times \tilde{\mathcal{V}}_a^\bullet) \circ U_a^\bullet = I_m^\bullet \times \tilde{\mathcal{U}}_a^\bullet,$$

with $I_m^\bullet = V_a^\bullet(m) \circ U_a^\bullet$ (the last identity follows from the \mathbf{A} -algebra homomorphism property of post-composition with U_a^\bullet). The conclusion follows from the fact that

$$I_m^\omega = 1 \text{ if } \omega = (m), \quad 0 \text{ if not.} \quad (12.6)$$

This formula can be checked by introducing the map $\rho_m: M^\bullet \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A}) \mapsto M_m^\bullet \in \mathcal{M}^\bullet(\Omega^*, \mathbf{A})$ defined by $M_m^\omega = M^\omega$ if $\|\omega\| = m$, 0 if not, and observing that $\rho_m(M^\bullet \circ U^\bullet) = \rho_m(M^\bullet) \circ U^\bullet$ for any licit mould U^\bullet ; thus $I_m^\bullet = \rho_m(V_a^\bullet) \circ U_a^\bullet = \rho_m(I_*^\bullet)$. \square

¹⁶This can be checked by means of the restriction homomorphism of the previous footnote: if U^\bullet is licit and U^ω is invertible whenever $r(\omega) = 1$, then any $U_0^\bullet \in \mathcal{M}^\bullet(\Omega, \mathbf{A})$ such that $\rho(U_0^\bullet) = U^\bullet$ and $U_0^{(0)} = 1$ is an invertible mould, the composition inverse of which has a restriction V^\bullet which satisfies $U^\bullet \circ V^\bullet = V^\bullet \circ U^\bullet = I_*^\bullet$; if moreover U^\bullet is alternal, then one can choose U_0^\bullet alternal (take $U_0^\omega = 0$ whenever $r(\omega) \geq 2$ and one of the letters of ω is 0), thus its inverse and the restriction of its inverse are alternal.

Remark 12.1. An analogous computation yields

$$(\partial + \nabla) \tilde{\mathcal{U}}_a^\bullet = -\tilde{\mathcal{U}}_a^\bullet \times \tilde{K}^\bullet$$

with a licit alternal mould $\tilde{K}^\bullet \in \mathcal{M}^\bullet(\mathbb{Z}^*, \mathbb{C}[[z^{-1}]])$ defined by $\tilde{K}^\omega = U_a^\omega \tilde{a}_{\|\omega\|}$ if $\|\omega\| \neq 0$.

12.4 As an application of the existence of Δ -friendly monomials, we now show

Theorem 8. Let $\mathbf{A} = \text{RES}_\mathbb{Z}^{\text{simp}}$. The subalgebra of $\text{End}_\mathbb{C} \mathbf{A}$ generated by the operators Δ_m , $m \in \mathbb{Z}^*$, is isomorphic to the free associative algebra on \mathbb{Z}^* .

In fact, we shall prove a stronger statement: for any non-commutative polynomial with coefficients in \mathbf{A} ,

$$P = \sum_{(m_1, \dots, m_r) \in \mathcal{F}} \tilde{\varphi}^{m_1, \dots, m_r} \Delta_{m_r} \cdots \Delta_{m_1}, \quad \mathcal{F} \text{ finite subset of } (\mathbb{Z}^*)^\bullet,$$

there exists $\tilde{\psi} \in \mathbf{A}$ such that $P\tilde{\psi} \neq 0$, unless all the coefficients $\tilde{\varphi}^{m_1, \dots, m_r}$ are zero. Thus there is no non-trivial polynomial relation between the alien derivations Δ_m .

Proof. Assume that not all the coefficients are zero. We may suppose $\mathcal{F} \neq \emptyset$ and $\tilde{\varphi}^\omega \neq 0$ for each $\omega \in \mathcal{F}$. Choose $\mathbf{m} = (m_1, \dots, m_r) \in \mathcal{F}$ with minimal length; then, for any family of Δ -friendly monomials $\tilde{\mathcal{U}}^\bullet$, we find $P\tilde{\mathcal{U}}^\mathbf{m} = \tilde{\varphi}^{m_1, \dots, m_r} \neq 0$ as a consequence of

$$\Delta_{m_s} \cdots \Delta_{m_1} \tilde{\mathcal{U}}^\omega = \begin{cases} \tilde{\mathcal{U}}^\mathbf{n} & \text{if } \omega = (m_1, \dots, m_s) \cdot \mathbf{n} \text{ with } \mathbf{n} \in (\mathbb{Z}^*)^\bullet, \\ 0 & \text{if not.} \end{cases} \quad (12.7)$$

□

12.5 Let us call *resurgence constant* any $\tilde{\varphi} \in \widetilde{\text{RES}}_\mathbb{Z}^{\text{simp}}$ such that $\Delta_m \tilde{\varphi} = 0$ for any $m \in \mathbb{Z}^*$. This is equivalent to saying that $\mathcal{B}\tilde{\varphi} = c\delta + \widehat{\varphi}(\zeta)$ with $c \in \mathbb{C}$ and $\widehat{\varphi}$ entire (in particular every convergent series $\tilde{\varphi}(z) \in \mathbb{C}\{z^{-1}\}$ is a resurgence constant, but the converse is not true since we did not require the Borel transform to be of exponential type: the entire function $\widehat{\varphi}$ might have order > 1).

Resurgence constants form a subalgebra $\widetilde{\mathcal{P}}_0$ of $\widetilde{\text{RES}}_\mathbb{Z}^{\text{simp}}$.

Proposition 12.1. Let $\tilde{\mathcal{U}}_1^\bullet$ and $\tilde{\mathcal{U}}_2^\bullet$ be two moulds in $\mathcal{M}^\bullet(\Omega^*, \widetilde{\text{RES}}_\mathbb{Z}^{\text{simp}})$ and suppose that $\tilde{\mathcal{U}}_1^\bullet$ is a family of Δ -friendly monomials. Then $\tilde{\mathcal{U}}_2^\bullet$ is a family of Δ -friendly monomials iff if there exists a symmetral mould $\widetilde{M}^\bullet \in \mathcal{M}^\bullet(\mathbb{Z}^*, \widetilde{\mathcal{P}}_0)$ such that

$$\tilde{\mathcal{U}}_2^\bullet = \tilde{\mathcal{U}}_1^\bullet \times \widetilde{M}^\bullet. \quad (12.8)$$

Thus all the families of Δ -friendly monomials can be deduced from one of them.

Proof. Let $\widetilde{M}^\bullet = (\widetilde{\mathcal{U}}_1^\bullet)^{-1} \times \widetilde{\mathcal{U}}_2^\bullet \in \mathcal{M}^\bullet(\Omega^*, \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}})$. This mould is symmetral iff $\widetilde{\mathcal{U}}_2^\bullet$ is symmetral. Let $m \in \mathbb{Z}^*$. We have $\Delta_m \widetilde{\mathcal{U}}_1^\bullet = I_m^\bullet \times \widetilde{\mathcal{U}}_1^\bullet$, with the mould I_m^\bullet defined by (12.6). The Leibniz rule applied to (12.8) yields $\Delta_m \widetilde{\mathcal{U}}_2^\bullet = I_m^\bullet \times \widetilde{\mathcal{U}}_2^\bullet + \widetilde{\mathcal{U}}_1^\bullet \times \Delta_m \widetilde{M}^\bullet$. Thus $\widetilde{\mathcal{U}}_2^\bullet$ satisfies (12.1) for all m iff $\widetilde{\mathcal{U}}_1^\bullet \times \Delta_m \widetilde{M}^\bullet = 0$ for all m , which is equivalent to $\widetilde{M}^\omega \in \widetilde{\mathcal{P}}_0$ since $\widetilde{\mathcal{U}}_1^\bullet$ admits a multiplicative inverse. \square

12.6 Define $\Delta_\emptyset = \text{Id}$ and $\Delta_\omega = \Delta_{\omega_r} \cdots \Delta_{\omega_1}$ for $\omega = (\omega_1, \dots, \omega_r) \in (\mathbb{Z}^*)^\bullet$. We call *resurgence polynomial* any $\tilde{\varphi} \in \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ such that $\Delta_\omega \tilde{\varphi} = 0$ for all but finitely many $\omega \in (\mathbb{Z}^*)^\bullet$. Resurgence polynomials form a subalgebra $\widetilde{\mathcal{P}}$ of $\widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ (which contains \mathcal{P}_0).

Proposition 12.2. *Let $\widetilde{\mathcal{U}}^\bullet$ be any family of Δ -friendly monomials and $\tilde{\varphi}$ be any simple resurgent function. Then $\tilde{\varphi}$ is a resurgence polynomial iff $\tilde{\varphi}$ can be written as*

$$\tilde{\varphi} = \sum_{\omega \in \mathcal{F}} \widetilde{\mathcal{U}}^\omega \tilde{\varphi}_\omega, \quad \mathcal{F} \text{ finite subset of } (\mathbb{Z}^*)^\bullet, \quad (12.9)$$

with $\tilde{\varphi}_\omega \in \widetilde{\mathcal{P}}_0$ for every $\omega \in \mathcal{F}$. Moreover, such a representation of a resurgence polynomial is unique and the formula $\mathcal{E} = \sum S \widetilde{\mathcal{U}}^\bullet \Delta_\bullet$ (with S defined by (5.3), thus $S \widetilde{\mathcal{U}}^\bullet$ is the multiplicative inverse of $\widetilde{\mathcal{U}}^\bullet$) defines an algebra homomorphism $\mathcal{E}: \widetilde{\mathcal{P}} \rightarrow \widetilde{\mathcal{P}}_0$ such that

$$\tilde{\varphi}_\omega = \mathcal{E} \Delta_\omega \tilde{\varphi}, \quad \omega \in (\mathbb{Z}^*)^\bullet.$$

Proof. In view of (12.7), formula (12.9) defines a resurgence polynomial whenever the $\tilde{\varphi}_\omega$'s are resurgence constants.

The formula $\mathcal{E} = \sum S \widetilde{\mathcal{U}}^\bullet \Delta_\bullet$ makes sense as an operator $\widetilde{\mathcal{P}} \rightarrow \widetilde{\text{RES}}_{\mathbb{Z}}^{\text{simp}}$ since the sum is locally finite; an easy adaptation of the arguments of Section 7 shows that \mathcal{E} is an algebra homomorphism because $\widetilde{\mathcal{U}}^\bullet$ is symmetral and Δ_\bullet can be viewed as a cosymmetral comould (the Δ_m 's which generate it are derivations of $\widetilde{\mathcal{P}}$).

Let us check that $\mathcal{E}(\widetilde{\mathcal{P}}) \subset \widetilde{\mathcal{P}}_0$. Let $\tilde{\varphi} \in \widetilde{\mathcal{P}}$ and $m \in \mathbb{Z}^*$; we can write $\Delta_m = \sum I_m^\bullet \Delta_\bullet$ with the notation (12.6). A computation analogous to the proof of Proposition 6.1, but taking into account the fact that Δ_m does not commute with the multiplication by $(S \widetilde{\mathcal{U}})^\omega$, shows that

$$\Delta_m \mathcal{E} \tilde{\varphi} = \sum ((S \widetilde{\mathcal{U}} \times I_m^\bullet) + \Delta_m S \widetilde{\mathcal{U}}^\bullet) \Delta_\bullet \tilde{\varphi}.$$

Since $\Delta_m \widetilde{\mathcal{U}}^\bullet = I_m^\bullet \times \widetilde{\mathcal{U}}^\bullet$ and S is an anti-homomorphism such that $SI_m^\bullet = -I_m^\bullet$ and $S\Delta_m = \Delta_m S$, we have $\Delta_m S \widetilde{\mathcal{U}}^\bullet = -S \widetilde{\mathcal{U}} \times I_m^\bullet$, hence $\Delta_m \mathcal{E} \tilde{\varphi} = 0$.

We conclude by considering $\tilde{\varphi} \in \widetilde{\mathcal{P}}$ and setting $\tilde{\varphi}_\alpha = \mathcal{E} \Delta_\alpha \tilde{\varphi}$ for every word $\alpha \in (\mathbb{Z}^*)^\bullet$ (but only finitely many words may yield a nonzero result). We have $\tilde{\varphi}_\alpha = \sum_\beta (S \widetilde{\mathcal{U}}^\bullet)^\beta \Delta_{\alpha \cdot \beta} \tilde{\varphi}$, thus $\sum_\alpha \widetilde{\mathcal{U}}^\alpha \tilde{\varphi}_\alpha = \sum_{(\alpha, \beta)} \widetilde{\mathcal{U}}^\alpha (S \widetilde{\mathcal{U}}^\bullet)^\beta \Delta_{\alpha \cdot \beta} \tilde{\varphi}$, and the identity $\widetilde{\mathcal{U}}^\bullet \times S \widetilde{\mathcal{U}}^\bullet = 1^\bullet$ implies $\sum_\alpha \widetilde{\mathcal{U}}^\alpha \tilde{\varphi}_\alpha = \tilde{\varphi}$. \square

13 Other applications of mould calculus

13.1 In this last section, we wish to indicate how mould calculus can be applied to another classical normal form problem: the linearisation of a vector field with non-resonant spectrum.

Let $\mathcal{A} = \mathbb{C}[[y_1, \dots, y_n]]$ with $n \in \mathbb{N}^*$, and consider a vector field with diagonal linear part:

$$X = \sum_{i=1}^n a_i(y) \frac{\partial}{\partial y_i}, \quad a_i(y) = \lambda_i y_i + \sum_{k \in \mathbb{N}^n, |k| \geq 2} a_{i,k} y^k$$

(with standard notations for the multi-indices: $y^k = y_1^{k_1} \cdots y_n^{k_n}$ and $|k| = k_1 + \cdots + k_n$ if $k = (k_1, \dots, k_n)$).

The first problem consists in finding a formal transformation which conjugates X and its linear part

$$X^{\text{lin}} = \sum_{i=1}^n \lambda_i y_i \frac{\partial}{\partial y_i}.$$

This linear part is thus considered as a natural candidate to be a normal form; it is determined by the spectrum $\lambda = (\lambda_1, \dots, \lambda_n)$. In fact $X^{\text{lin}} = \mathcal{X}_\lambda$ with the notation (6.4).

It is not always possible to find a formal conjugacy between X and X^{lin} , because elementary calculations let appear rational functions of the spectrum, the denominators of which are of the form

$$\langle m, \lambda \rangle = m_1 \lambda_1 + \cdots + m_n \lambda_n \tag{13.1}$$

with certain multi-indices $m \in \mathbb{Z}^n$. Let us make the following *strong non-resonance assumption*:

$$\langle m, \lambda \rangle \neq 0 \quad \text{for every } m \in \mathbb{Z}^n \setminus \{0\}. \tag{13.2}$$

We shall now indicate how to construct a formal conjugacy via mould-comould expansions under this assumption.

13.2 We are in the framework of Section 6 with $\mathbf{A} = \mathbb{C}$. Let us use the standard monomial valuation on \mathcal{A} , defined by $\nu(y^k) = |k|$. We shall manipulate operators of \mathcal{A} having a valuation with respect to ν ; they form a subspace \mathcal{F} of $\text{End}_{\mathbb{C}} \mathcal{A}$ which was denoted $\mathcal{F}_{\mathcal{A}, \mathbf{A}}$ in (6.2).

We first decompose X as a sum of homogeneous components, in the sense of Definition 6.2: X^{lin} is homogeneous of degree 0 and we can write

$$X - X^{\text{lin}} = \sum_{i=1}^n \sum_{k \in \mathbb{Z}^n} a_{i,k} y^k \frac{\partial}{\partial y_i},$$

thus extending the definition of the $a_{i,k}$'s:

$$a_{i,k} \neq 0 \Rightarrow k \in \mathbb{N}^n \text{ and } |k| \geq 2.$$

Using the canonical basis (e_1, \dots, e_n) of \mathbb{Z}^n , we can write

$$X - X^{\text{lin}} = \sum_{m \in \mathbb{Z}^n} B_m, \quad B_m = \sum_{i=1}^n a_{i,m+e_i} y^m \cdot y_i \frac{\partial}{\partial y_i}. \quad (13.3)$$

Observe that each B_m is homogeneous of degree $m \in \mathbb{Z}^n$ and that

$$B_m \neq 0 \Rightarrow m \in \mathcal{N},$$

$$\mathcal{N} = \{ m \in \mathbb{Z}^n \mid \exists i \text{ such that } m + e_i \in \mathbb{N}^m \text{ and } |m| \geq 1 \}.$$

We thus view \mathcal{N} as an alphabet and consider

$$\mathbf{B}_\emptyset = \text{Id}, \quad \mathbf{B}_{m_1, \dots, m_r} = B_{m_r} \cdots B_{m_1}$$

as a comould on \mathcal{N} with values in \mathcal{F} . For instance, $X - X^{\text{lin}} = \sum I^\bullet \mathbf{B}_\bullet$. The inequalities

$$\text{val}_\nu(\mathbf{B}_{m_1, \dots, m_r}) \geq |m_1 + \cdots + m_r|$$

show that, for any scalar mould $M^\bullet \in \mathcal{M}^\bullet(\mathcal{N}, \mathbb{C})$, the family $(M^{\mathbf{m}} \mathbf{B}_\mathbf{m})_{\mathbf{m} \in \mathcal{N}^\bullet}$ is formally summable in \mathcal{F} (indeed, for any $\delta \in \mathbb{Z}$, $\text{val}_\nu(M^{m_1, \dots, m_r} \mathbf{B}_{m_1, \dots, m_r}) \leq \delta$ implies $r \leq |m_1| + \cdots + |m_r| \leq \delta$ and there are only finitely many $\eta \in \mathcal{N}$ such that $|\eta| \leq \delta$).

13.3 According to the general strategy of mould-comould expansions, we now look for a formal conjugacy θ between X and X^{lin} through its substitution automorphism Θ , which should satisfy $X = \Theta^{-1} X^{\text{lin}} \Theta$. This conjugacy equation can be rewritten

$$[X^{\text{lin}}, \Theta] = \Theta(X - X^{\text{lin}}).$$

Propositions 6.1 and 6.3 show that, given any $M^\bullet \in \mathcal{M}^\bullet(\mathcal{N}, \mathbb{C})$, $\Theta = \sum M^\bullet \mathbf{B}_\bullet$ is solution as soon as

$$D_\varphi M^\bullet = I^\bullet \times M^\bullet, \quad (13.4)$$

with $D_\varphi M^{\mathbf{m}} = \langle \|\mathbf{m}\|, \lambda \rangle M^{\mathbf{m}}$ for $\mathbf{m} \in \mathcal{N}^\bullet$.

Assumption (13.2) allows us to find a unique solution of equation (13.4) such that $M^\emptyset = 1$; it is inductively determined by

$$M^{m_1, \dots, m_r} = \frac{1}{\langle \|\mathbf{m}\|, \lambda \rangle} M^{m_2, \dots, m_r},$$

hence

$$M^{\mathbf{m}} = \frac{1}{\langle m_1 + \cdots + m_r, \lambda \rangle} \frac{1}{\langle m_2 + \cdots + m_r, \lambda \rangle} \cdots \frac{1}{\langle m_r, \lambda \rangle} \quad (13.5)$$

The symmetrality of this solution can be obtained by mimicking the proof of Proposition 5.5.

Since \mathbf{B}_\bullet is cosymmetrical, we thus have an automorphism $\Theta = \sum M^\bullet \mathbf{B}_\bullet$; since Θ is continuous for the Krull topology, $\theta = (\theta_1, \dots, \theta_n)$ with $\theta_i = \Theta y_i$ yields a formal tangent-to-identity transformation which conjugates X and X^{lin} .

13.4 As was alluded to at the end of Section 7, the formalism of moulds can be equally applied to the normalisation of discrete dynamical systems. A problem

parallel to the previous one is the linearisation of a formal transformation with multiplicatively non-resonant spectrum.

Suppose indeed that $f = (f_1, \dots, f_n)$ is a n -tuple of formal series of \mathcal{A} without constant terms, with diagonal linear part $f^{\text{lin}}: (y_1, \dots, y_n) \mapsto (\ell_1 y_1, \dots, \ell_n y_n)$. Conjugating f and f^{lin} is equivalent to finding a continuous automorphism Θ which conjugates the corresponding substitution automorphisms: $F = \Theta^{-1} F^{\text{lin}} \Theta$.

This is possible under the following *strong multiplicative non-resonance assumption* on the spectrum $\ell = (\ell_1, \dots, \ell_n)$:

$$\ell^m - 1 \neq 0 \quad \text{for every } m \in \mathbb{Z}^n \setminus \{0\}. \quad (13.6)$$

An explicit solution is obtained by expanding $F(F^{\text{lin}})^{-1}$ in homogeneous components

$$F = \left(\text{Id} + \sum_{m \in \mathcal{N}} B_m \right) F^{\text{lin}},$$

where the homogeneous operators B_m are no longer derivations; instead, they satisfy the modified Leibniz rule (7.4) and generate a *cosymmetrel* comould \mathbf{B}_\bullet . Correspondingly, the scalar mould

$$M^\emptyset = 1, \quad M^{\mathbf{m}} = \frac{1}{(\ell^{m_1+\dots+m_r} - 1)(\ell^{m_2+\dots+m_r} - 1) \cdots (\ell^{m_r} - 1)} \quad (13.7)$$

is *symmetrel* and $\Theta = \sum M^{\bullet} \mathbf{B}_\bullet$ is the desired automorphism (see [7]), whence a formal tangent-to-identity transformation θ which conjugates f and f^{lin} .

13.5 In both previous problems, it is a classical result that a formal linearising transformation θ exists under a weaker non-resonance assumption: namely, it is sufficient that (13.2) or (13.6) hold with $\mathbb{Z}^n \setminus \{0\}$ replaced by $\mathcal{N} \setminus \{0\}$. Unfortunately, this is not clear on the mould-comould expansion, since under this weaker assumption the formula (13.5) or (13.7) may involve a zero divisor, thus the mould M^\bullet is not well-defined.

J. Écalle has invented a technique called *arborification* which solves this problem and which goes far beyond: arborification also allows to recover the Bruno-Rüssmann theorem, according to which the formal linearisation θ is convergent whenever the vector field X or the transformation f is convergent and the spectrum λ or ℓ satisfies the so-called *Bruno condition* (a Diophantine condition which states that the divisors $\langle m, \lambda \rangle$ or $\ell^m - 1$ do not approach zero “abnormally well”).

The point is that, even when X or f is convergent and the spectrum is Diophantine, it is hard to check that $\theta_i(y)$ is convergent because it is represented as the sum of a formally summable family $(M^{\mathbf{m}} \mathbf{B}_\bullet y_i)_{\mathbf{m} \in \mathcal{N}^\bullet}$ in $\mathbb{C}[[y_1, \dots, y_n]]$, but the family $(|M^{\mathbf{m}} \mathbf{B}_\bullet y_i|)_{\mathbf{m} \in \mathcal{N}^\bullet}$ may fail to be summable in \mathbb{C} for any $y \in \mathbb{C}^n \setminus \{0\}$. However, arborification provides a systematic way of reorganizing the terms of the sum: $\theta_i(y)$ then appears as the sum of a summable family indexed by “arborescent sequences” rather than words. The reader is referred to [5], [6], [11], and also to the recent article [10].

13.6 There is another context, totally different, in which J. Écalle has used mould calculus with great efficiency. The multizeta values

$$\zeta(s_1, s_2, \dots, s_r) = \sum_{n_1 > n_2 > \dots > n_r > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}$$

naturally present themselves as a scalar mould on \mathbb{N}^* ; in fact,

$$\zeta(s_1, \dots, s_r) = \text{Ze}^{\left(\begin{smallmatrix} 0, & \dots, & 0 \\ s_1, & \dots, & s_r \end{smallmatrix}\right)},$$

with $\text{Ze}^{\left(\begin{smallmatrix} \varepsilon_1, & \dots, & \varepsilon_r \\ s_1, & \dots, & s_r \end{smallmatrix}\right)} = \sum_{n_1 > \dots > n_r > 0} \frac{e^{2\pi i(n_1\varepsilon_1 + \dots + n_r\varepsilon_r)}}{n_1^{s_1} \cdots n_r^{s_r}}$

for $s_1, \dots, s_r \in \mathbb{N}^*$, $\varepsilon_1, \dots, \varepsilon_r \in \mathbb{Q}/\mathbb{Z}$ (with a suitable convention to handle possible divergences). The mould Ze^\bullet is the central object; it turns out that it is *symmetrel*. It is called a *bimould* because the letters of the alphabet are naturally given as members of a product space, here $\mathbb{N}^* \times (\mathbb{Q}/\mathbb{Z})$; this makes it possible to define new operations and structures. This is the starting point of a whole theory, aimed at describing the algebraic structures underlying the relations between multizeta values. See [8] or [9].

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